Chapter 9
Exercices Week 7 (Chapter 9+11)

Chapitre 9 and 11

9.1. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Let π_1, π_2 two invariant probability measures for P.

1. Show that $\pi_1 \wedge \pi_2$ is an invariant measure for P.
2. Deduce that there exists two invariant probability measures for P that are mutually singular.
3. If P is ϕ-irreducible, show that π_1 dominates ϕ.
4. Deduce that if P is irreducible, there is at most one invariant probability measure.

9.2. Let $\lambda \in \mathbb{M}_+(\mathscr{X})$ and $\eta \in (0,1)$. Then show that λ is invariant for P if and only if it is invariant for K_{a_η}.

9.3. Let C be an accessible small set.

1. Show that there exists an integer m and a measure μ_0 such that C is (m,μ_0)-small set with $\mu_0(C) > 0$.
2. Show that C is an accessible $(1,\mu)$-small set with $\mu(C) > 0$ for the resolvent kernel K_{a_η} for any $\eta > 0$, i.e. C is strongly aperiodic for the resolvent kernel.
3. Show that $\sum_{n=1}^{\infty} K_{a_\eta}^n = \frac{1-\eta}{\eta} U$.
4. Deduce that if C is recurrent for P, then it is also recurrent for K_{a_η}.

9.4. Let C be a $(1,\varepsilon\nu)$-small set for P. Assume that P is irreducible and that $M := \sup_{x\in\mathsf{X}} \mathbb{E}[\sigma_C] < \infty$. Define $\check{\alpha} = C \times \{1\}$ and $\check{\mathsf{X}} = \mathsf{X} \times \{0,1\}$ and $\check{\mathscr{X}} = \mathscr{X} \otimes \mathscr{P}(\{0,1\})$. Define by $\check{\mathbb{P}}_{\check{\xi}}$ the probability induced on the canonical space $(\check{\mathsf{X}}^{\mathbb{N}}, \check{\mathscr{X}}^{\otimes\mathbb{N}})$ by the split kernel $\check{P}$, starting from the probability measure $\check{\xi}$ on $(\check{\mathsf{X}}, \check{\mathscr{X}})$.

1. Define $r_k = \check{\mathbb{E}}_{\check{\alpha}}\left[\sigma_C^k \mathbb{1}\left\{d_{X_{\sigma_C^\ell}} = 0,\ \forall \ell \in [1:k-1]\right\}\right]$ where σ_C^ℓ are the successive visits of (X_n) to the set C. Show that $r_k \le r_{k-1}(1-\varepsilon) + M(1-\varepsilon)^{k-1}$.
2. Deduce that $\check{\mathbb{E}}_{\check{\alpha}}[\sigma_{\check{\alpha}}] < \infty$.