Chap 18 Uniform ergodicity & V-geometric ergodicity.

Th 18.1.1 (P^t fixe).

Hyp: Let P M.K on $\mathcal{X} \times \mathcal{X}$. Let $\mathbb{F}$ subspace de $M_1(\mathcal{X})$, ρ metric sur $\mathbb{F}$ s.t. $(\mathbb{F}, \rho)$ complete. Moreover $\boxed{\forall x \in \mathcal{X},\ \delta_x \in \mathbb{F}}$.

We assume: $\exists\ A_r \geq 0\ (r \in \{1, .., m-1\})$, $\alpha \in [0,1[$ s.t $\forall \xi, \xi' \in \mathbb{F}$.

$$\rho(\xi P^r, \xi' P^r) \leq A_r \rho(\xi, \xi'), \quad r \in \{1, .., m-1\}. \qquad (*)$$

$$\rho(\xi P^m, \xi' P^m) \leq \alpha \rho(\xi, \xi'). \qquad (**)$$

Then, $\exists!$ invariant prob. measure $\pi \in \mathbb{F}$ et $\forall \xi \in \mathbb{F}$, $\forall n \geq 0$,

$$(***) \quad \rho(\xi P^n, \pi) \leq \left(1 \vee \max_{1 \leq r \leq m-1} A_r\right) \rho(\xi, \pi)\, \alpha^{\lfloor n/m \rfloor}$$

Moreover, if: we have $\left(\rho(\underset{\in F}{\mu_n}, \underset{\in F}{\mu}) \to 0\right) \to \left(\forall f \text{ mesurable bornée}, \mu_n(f) \to \mu(f)\right)$

or $\left(\quad\text{————}\quad\right) \Rightarrow \left(\forall f\ C^0\text{-bornée}, \mu_n(f) \to \mu(f)\right)$ (si $\mathcal{X}$ is a metric space $\mathcal{X}$ Borel sigma field.)

then π is the unique P-invariant measure in $M_1(\mathcal{X})$.

Proof: . If $\pi = \pi P$, $\pi' = \pi' P$ where $\pi, \pi' \in \mathbb{F}$.

then: $\rho(\underbrace{\pi P^m}_{\pi}, \underbrace{\pi' P^m}_{\pi'}) \leq \alpha \rho(\pi, \pi')$. (par $(**)$)

$$\rho(\pi, \pi') \leq \underbrace{\alpha}_{\in [0,1[} \rho(\pi, \pi') \Rightarrow \rho(\pi, \pi') = 0 \Rightarrow \boxed{\pi = \pi'}.$$

This shows the uniqueness of an invariant prob. meas. in $\mathbb{F}$.

. $\forall n \in \mathbb{N}$, $n = m \left\lfloor \frac{n}{m} \right\rfloor + r$. where $r \in [\![0, m-1]\!]$

$\forall \xi, \xi' \in \mathbb{F}$,
$$\rho(\xi P^n, \xi' P^n) = \rho\left(\xi P^r (P^m)^{\lfloor \frac{n}{m} \rfloor}, \xi' P^r (P^m)^{\lfloor \frac{n}{m} \rfloor}\right).$$

$$\leq \alpha^{\lfloor \frac{n}{m} \rfloor} \rho(\xi P^r, \xi' P^r)^{\leq (A_r \rho(\xi, \xi') \mathbb{1}(r \in [\![1, m-1]\!]) + \rho(\xi, \xi') \mathbb{1}(r=0)).}$$

$$\forall \xi, \xi' \in \mathbb{F}, \rho(\xi P^n, \xi' P^n) \leq \alpha^{\lfloor \frac{n}{m} \rfloor} \left(\max_{i : [\![1, m-1]\!]} A_i \vee 1\right) \rho(\xi, \xi'). \qquad (****)$$

Set $A_0 = 1$.

Set $\xi' = \xi P$

$$\rho(\xi P^n, \xi P^{n+1}) \leq \alpha^{\lfloor \frac{n}{m} \rfloor} \max_{i \in [\![0, m-1]\!]} A_i \quad \rho(\xi, \xi P).$$

then, (ξP^n) is a Cauchy sequence $(\%\ \rho)$ and $(\mathbb{F}, \rho)$ is complete.

Then, $\exists\, \pi \in \mathbb{F}$, ξ: $\rho(\xi P^n, \pi) \underset{n \to +\infty}{\longrightarrow} 0$.

$$\rho(\xi P^{n+1}, \pi P) \leq A_1 \underbrace{\rho(\xi P^n, \pi)}_{\to 0}. \qquad \rho(\xi P^{n+1}, \pi P) \to 0.$$

then: $\boxed{\pi = \pi P}$.

We set: $\xi' = \pi$ in $(\ast\ast\ast\ast)$ and we obtain $(\ast\ast\ast)$

Last part: Let $\tilde{\pi}$ invariant prob. meas. (ie: in $M_1(\mathcal{X})$)

then: $\forall f \in \mathbb{F}_b(\mathcal{X})$. (measurable and bounded).

$$\tilde{\pi}(f) = \int \underbrace{P^n f(x)}_{\substack{\to \pi(f) \\ \leq \sup|f|}} \tilde{\pi}(dx). \qquad \text{but } \delta_x \in \mathbb{F}, \ \rho(\delta_x P^n, \pi) \underset{n \to +\infty}{\longrightarrow} 0.$$

$$\Rightarrow P^n f(x) \underset{n \to +\infty}{\longrightarrow} \pi f.$$

$\tilde{\pi}(f) \underset{n \to +\infty}{\longrightarrow} \pi(f)$. ie: $\tilde{\pi}(f) = \pi(f)$. ie: $\boxed{\tilde{\pi} = \pi}$.

Def: 19.2.1 (Dobrushin coeff.) Let P be a MK on $\mathcal{X} \times \mathcal{X}$.

$\Delta(P)$ = Dobrushin coeff. is the lipschitz coeff. of P % total variation distance:

$$\Delta(P) = \sup_{\xi \neq \xi' \in M_1(\mathcal{X})} \frac{d_{TV}(\xi P, \xi' P)}{d_{TV}(\xi, \xi')} = \sup_{\xi \neq \xi' \in M_1(\mathcal{X})} \frac{\|\xi P - \xi' P\|_{TV}}{\|\xi - \xi'\|_{TV}}.$$

$$|\xi - \xi'|(A) = \int_A |f - g|\, d\mu.$$

Rappel: $\cdot\ d_{TV}(\xi, \xi') = \dfrac{\|\xi - \xi'\|_{TV}}{2}. \quad \in [0,1].$

$$d\xi = f\, d\mu, \quad d\xi' = g\, d\mu.$$

$$\|\xi - \xi'\|_{TV} = \underbrace{\sup\{|\xi h - \xi' h|;\ |h| \leq 1\}}_{A} = \underbrace{\int |f - g|\, d\mu}_{B} = |\xi - \xi'|(\mathcal{X})$$

$$= (\xi - \xi')^+(\mathcal{X}) + (\xi' - \xi)^+(\mathcal{X}) = 2(\xi - \xi')^+(\mathcal{X})$$

$A \leq B$: $|\xi h - \xi' h| = \left|\int (f - g) h\, d\mu\right| \leq \int |f - g||h|\, d\mu \leq B$.

$\underline{B \leq A}$. $\int |f-g|\, d\mu = \int (f-g) \underbrace{sg(f-g)}_{h \in [-1,1]} d\mu = \xi(h) - \xi'(h) \leq A.$

<u>Thm:</u>

$\Delta(PQ) \leq \Delta(P)\Delta(Q)$.

<u>Lemma: 18.2.2</u> $\underbrace{\Delta(P)}$ $= \underbrace{\sup_{(x,x')\in X^2} d_{TV}(P(x,\cdot), P(x',\cdot))}_{C} \leq 1.$

<u>Dém:</u>

On a clairement : $\boxed{C \leq \Delta(P).}$

Soit $\xi, \xi' \in M_1(X)$, $\xi \neq \xi'$.

$$\|\xi P - \xi' P\|_{TV} = \sup_{osc(h)\leq 2} |\xi Ph - \xi' Ph|.$$

$$= \sup_{osc(h)\leq 2} \underbrace{\left| \int [\xi(dx) - \xi'(dx)]\, Ph(x) \right|}.$$

$$= \sup_{osc(h)\leq 2} \left| \int (\xi-\xi')^+(dx)\, Ph(x) - \int (\xi'-\xi)^+(dx')\, Ph(x') \right|.$$

$$1 = \frac{\int (\xi'-\xi)^+(dx')}{(\xi'-\xi)^+(X)} \qquad \underbrace{\qquad}_{=} \qquad \frac{\int (\xi-\xi')^+(dx)}{(\xi-\xi')^+(X)}$$

$$= \sup_{osc(h)\leq 2} \left| \iint_{X^2} \frac{(\xi-\xi')^+(dx)\,(\xi'-\xi)^+(dx')}{(\xi'-\xi)^+(X)} \left(\underbrace{Ph(x) - Ph(x')} \right) \right|.$$

$$\leq \underbrace{\sup_{osc(h)\leq 2} \left(\sup_{x\neq x'} |\underbrace{Ph(x)}_{\delta_x Ph} - \underbrace{Ph(x')}_{\delta_{x'} Ph}| \right)} \cdot (\xi'-\xi)^+(X).$$

$$\sup_{x\neq x'} \underbrace{\sup_{osc\, h\leq 2} \frac{|Ph(x) - Ph(x')|}{2}}. \quad \|\xi'-\xi\|_{TV}$$

$$\sup_{x\neq x'} d_{TV}(\delta_x P, \delta_{x'} P).$$

Ainsi : $\forall \xi \neq \xi'$. $\dfrac{\|\xi P - \xi' P\|_{TV}}{\|\xi - \xi'\|_{TV}} \leq \sup_{x\neq x'} d_{TV}(P(x,\cdot), P(x',\cdot)) = C$.

Donc : $\boxed{\Delta(P) \leq C}$

Finally : $\boxed{\Delta(P) = C.}$

Lemma 18.2.3 $\forall \xi, \xi' \in M_1(\mathcal{X})$, $d_{TV}(\xi P^n, \xi' P^n) \downarrow$. (because $\Delta(P) \leq 1$)

and: $d_{TV}(\xi P^n, \xi' P^n) \leq \Delta(P)^n d_{TV}(\xi, \xi')$.

In particular, if $\pi = \pi P$.

$$d_{TV}(\xi P^n, \pi) \leq \Delta(P)^n d_{TV}(\xi, \pi). \quad \text{and} \quad d_{TV}(\xi P^n, \pi) \downarrow.$$

(Ergod. unif). Thm 18.2.4 : if $\Delta(P^m) \leq 1-\varepsilon$. then: P admits a uniq. inv. prob. meas. π and: $\forall \xi \in M_1(\mathcal{X})$

$$\|\xi P^n - \pi\|_{TV} \leq (1-\varepsilon)^{\lfloor \frac{n}{m} \rfloor} \|\xi - \pi\|_{TV}.$$

Proof: Apply: thm 18.1.1. with: $\rho = d_{TV}$.

Def 18.2.5 (Doeblin set and uniform Doeblin condition)

A set $C \in \mathcal{X}$ is an (m, ε)-Doeblin set if $\forall (x, x') \in C \times C$,

$$d_{TV}(P^m(x, \cdot), P^m(x', \cdot)) \leq (1-\varepsilon)$$

(Rq: the r.h.s. is: $(1-\varepsilon) \underbrace{d_{TV}(\delta_x, \delta_{x'})}_{=1}$.).

if: $\mathcal{X}$ is a Doeblin set., we say that P^m satisfies a uniform Doeblin condition.

18.3 V-Dobrushin coefficient.

If $f \in F(\mathcal{X})$. (f measurable). $|f|_V = \sup_{x \in \mathcal{X}} \frac{|f(x)|}{V(x)}$ where $V \geq 1$.

$$\mathrm{osc}_V(f) = \sup_{x, x'} \frac{|f(x) - f(x')|}{V(x) + V(x')}.$$

$\mathrm{osc}_{(1/2)}(f) = \mathrm{osc}(f)$.
i.e.: $\mathrm{osc}_1(f) = \frac{\mathrm{osc}(f)}{2}$

if $\xi \in M_b(\mathcal{X})$. (bounded signed measure), ($\|\xi\|_V = |\xi|(V)$.).

$d_V(\xi, \xi') = \frac{1}{2} \|\xi - \xi'\|_V$., , $\|\xi - \xi'\|_V = \sup(\{(\xi - \xi')h;\ \mathrm{osc}_V(h) \leq 1\})$

$= \sup\{(\xi - \xi')h,\ |h|_V \leq 1\}$

$\|\xi - \xi'\|_V = \int |f - g|(x) V(x) d\mu$.

(where: $|f|_V = \sup_x |f(x)| / V(x)$.)

$\left(h(x) = \operatorname{sg}(f-g)(x) \times V(x).\right)$

$M_{1,V}(\mathcal{X}) = \{ \xi \in M_1(\mathcal{X}), \ \xi(V) < \infty \}$, we admit that $(M_{1,V}(\mathcal{X}), d_V)$ is a complete metric space.

$\boxed{\text{Lemme 11.3.1}}$: $\forall \xi, \xi' \in M_1(\mathcal{X})$, if $d_{1,V}(\xi, \xi') \le 1 - \varepsilon$ then: $\|\xi - \xi'\|_V \le \xi(V) + \xi'(V) - 2\varepsilon$.

$\left| \begin{array}{l} d\xi = f\, d\mu \\ d\xi' = g\, d\mu \end{array} \right.$

Dém :
$$\|\xi - \xi'\|_V = \int |f-g| V \, d\mu = \int (f \vee g - f \wedge g) \, V \, d\mu$$

$$\underbrace{f\vee g + f\wedge g}_{= f+g} - 2 f \wedge g$$

$$= \int f V d\mu + \int g V d\mu - 2 \int (f \wedge g) V d\mu$$
$$\le \xi(V) + \xi'(V) - 2 \underbrace{\int (f \wedge g)\, d\mu}_{\ge \varepsilon}$$

En effet : $\frac{1}{2} \int |f-g| \, d\mu \le 1 - \varepsilon$

$\Leftrightarrow \varepsilon \le 1 - \frac{1}{2} \int |f-g| \, d\mu = 1 - \frac{1}{2} \int (f+g-2f\wedge g) \, d\mu$

$= 1 - \frac{1}{2}\left(1 + 1 - 2\int f \wedge g \, d\mu\right) = \int (f \wedge g)\, d\mu$

V-Doeb. coeff :
$$\Delta_V(P) = \overbrace{\sup_{\xi \ne \xi' \in M_{1,V}(\mathcal{X})} \underbrace{\frac{d_V(\xi P, \xi' P)}{d_V(\xi, \xi')}}_{\frac{\|\xi P - \xi' P\|_V}{\|\xi - \xi'\|_V}}}^{\Delta_V(P)} = \overbrace{\sup_{x \ne x'} \frac{\|P(x,\cdot) - P(x',\cdot)\|_V}{V(x) + V(x')}}^{A}$$
$$= \sup_{\operatorname{osc}_V(f) \le 1} \operatorname{osc}_V(Pf).$$

Preuve

Il suffit de mq : $\Delta_V(P) \le A$

$$\|\xi P - \xi' P\|_V \overset{?}{\le} A \|\xi - \xi'\|_V \qquad (*)$$

$(\xi - \xi')^+(dx) - (\xi' - \xi)^+(dx)$

$$\|\xi P - \xi' P\|_V = \sup_{\operatorname{osc}_V(h) \le 1} |\xi P h - \xi' P h| = \sup_{\operatorname{osc}_V(h) \le 1} \left| \int_X (\xi - \xi')(dx) \, Ph(x) \right|$$

$$= \sup_{\operatorname{osc}_V(h) \le 1} \left| \int (\xi - \xi')^+(dx) Ph(x) \frac{\int (\xi' - \xi)^+(dx')}{(\xi' - \xi)(X)} - \int (\xi' - \xi)^+(dx') Ph(x') \frac{\int (\xi - \xi')^+(dx)}{(\xi - \xi')^+(X)}\right.$$

$$= \sup_{\operatorname{osc}_V(h)\leq 1.} \left| \iint_{\mathcal{X}^2} \frac{(\xi-\xi')^+(dx)\,.\,(\xi'-\xi)^+(dx')}{(\xi'-\xi)^+(\mathcal{X})} \, (V(x)+V(x'))\,.\left(\frac{Ph(x)-Ph(x')}{V(x)+V(x')}\right). \right|$$

$$\leq \sup_{\operatorname{osc}_V(h)\leq 1.} \sup_{x\neq x'.} \frac{|Ph(x)-Ph(x')|}{V(x)+V(x')} \times \Bigg| \underbrace{\int (\xi-\xi')^+(dx)\,V(x) + \int (\xi'-\xi)^+(dx')\,V(x')}_{\underbrace{|\xi-\xi'|(V)}_{\|\xi-\xi'\|_V.}} \Bigg|$$

$$\leq \underbrace{\sup_{x\neq x'.} \sup_{\operatorname{osc}_V(h)\leq 1.} \left|\frac{Ph(x)-Ph(x')}{V(x)+V(x')}\right|}_{\underbrace{\leq \sup_{x\neq x'.} \frac{\|P(x,\cdot)-P(x',\cdot)\|_V}{V(x)+V(x')}}_{A}} \times \|\xi-\xi'\|_V.$$

That shows (*). and the proof is completed.

Lemma 11.4.2 $\exists V: \mathcal{X}\to[1,\infty[$, $\exists \lambda\in]0,1[$, $b\in\mathbb{R}^+$; $d\in\mathbb{R}^+$.

Hyp: (i) $C=\{V\leq d\}$ is a $(1,\varepsilon)$ Doeblin set $\left(\forall x,x'\in C,\ \|P(x,\cdot)-P(x',\cdot)\|_{TV}\leq 1-\varepsilon\right)$

(ii) $PV\leq \lambda V+b$.

(iii) $\tilde\lambda = \lambda + \dfrac{2b}{1+d} < 1$.

Then: $\exists \beta\in]0,1[$, $\Delta_{[\underbrace{1-\beta+\beta V}_{V_\beta.}]}(P) < 1.$

$$\frac{\|P(x,\cdot)-P(x',\cdot)\|_{TV}}{\|\delta_x-\delta_{x'}\|_{TV}=2} \leq \begin{cases} \boxed{1-\varepsilon} & \text{si } (x,x')\in C\times C. \\ \boxed{1} & \text{si } (x,x')\notin C\times C. \end{cases}$$

$$\frac{\|P(x,\cdot)-P(x',\cdot)\|_V}{V(x)+V(x')} \leq \begin{cases} \boxed{\dfrac{PV(x)+PV(x')-2\varepsilon}{V(x)+V(x')}} & \text{si } (x,x')\in C\times C. \\ \underbrace{\dfrac{PV(x)+PV(x')}{V(x)+V(x')}}_{\leq \lambda\frac{(V(x)+V(x'))+2b}{V(x)+V(x')}\leq \lambda+\frac{2b}{V(x)+V(x')}.\leq \boxed{\lambda+\frac{2b}{1+d}}<1.} & \text{si } (x,x')\notin C\times C. \end{cases}$$

$\tilde\lambda$

car si $(x,x') \notin C\times C$, $\underbrace{V(x)+V(x')}\geq 1+d$.

$V_\beta = \underbrace{(1-\beta)} + \beta \underbrace{V}$.

by 18.3.1.

$$\frac{\|P(x,\cdot)-P(x',\cdot)\|_{V_\beta}}{V_\beta(x)+V_\beta(x')} \leq \frac{PV_\beta(x)+PV_\beta(x')-2\varepsilon \mathbb{1}_{C\times C}(x,x')}{V_\beta(x)+V_\beta(x')}$$

$$\leq \frac{2(1-\beta)+\beta\left(PV(x)+PV(x')\right)-2\varepsilon\mathbb{1}_{C\times C}(x,x')}{V_\beta(x)+V_\beta(x')} = A_\beta(x,x')$$

$u \longmapsto \frac{a+bu}{c+du}$ monotone

Cas 1 $(x,x') \notin C\times C$.

$$A_\beta(x,x') \leq \frac{2(1-\beta)+\beta\tilde{\lambda}\left(V(x)+V(x')\right)}{V_\beta(x)+V_\beta(x')} = \frac{2(1-\beta)+\beta\tilde{\lambda}\left(V(x)+V(x')\right)}{2(1-\beta)+\beta\left(\underbrace{V(x)+V(x')}\right)}$$

$$\leq \underbrace{\frac{2(1-\beta)+\beta\tilde{\lambda}(1+d)}{2(1-\beta)+\beta(1+d)}}_{<1 \ (\forall \beta\in]0,1[} \vee \underbrace{\tilde{\lambda}}_{<1}.$$

Cas 2: $(x,x')\in C\times C$.

$2 \leq V(x)+V(x') \leq 2d$.

$$A_\beta(x,x') \leq \frac{2(1-\beta)+\beta\left(\lambda\left(V(x)+V(x')\right)+2b\right)-2\varepsilon}{\underbrace{2(1-\beta)+\beta\left(V(x)+V(x')\right)}_{H_\beta(V(x)+V(x'))}}$$

$$\leq \underbrace{H_\beta(2)}_{\xrightarrow[\beta\to 0]{} 1-\varepsilon} \vee \underbrace{H_\beta(2d)}_{\xrightarrow[\beta\to 0]{} 1-\varepsilon}.$$