Alain Oliviero-Durmus alain.durmus@polytechnique.edu
École Polytechnique
Stochastic calculus for machine learning, Master MVA

9 Stochastic Differential Equation

Exercise 9.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, B)$ be a filtered probability space carrying a real Brownian motion $(B_t)_{t\geq 0}$ started from 0. Let σ and b be Borel functions on $\mathbb{R}$ such that, for $M, K \geq 0$ it holds: for all $x, y \in \mathbb{R}$,

$$|\sigma(x)| \leq M \;, \qquad |b(x)| \leq M \;, \qquad |\sigma(x) - \sigma(y)| \leq K|x-y| \;, \qquad |b(x) - b(y)| \leq K|x-y| \;.$$

Fix $x \in \mathbb{R}$, and let $(X_t)_{t\geq 0}$ be the unique solution of

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_0^t b(X_s)\,\mathrm{d}s \;.$$

For $n \geq 1$, define the Euler scheme $(X_t^n)_{t\geq 0}$ by $X_0^n = x$ and

$$X_t^n = X_{k/n}^n + \sigma(X_{k/n}^n)(B_t - B_{k/n}) + b(X_{k/n}^n)\,(t - k/n) \;\; , \; t \in [k/n, (k+1)/n) \;\; .$$

Also define

$$\tau_s^n = \sum_{k=0}^{\infty} \frac{k}{n}\,\mathbb{1}_{[k/n,(k+1)/n)}(s) \;.$$

(1) Show that

$$X_t^n = x + \int_0^t \sigma(X_{\tau_s^n}^n)\,\mathrm{d}B_s + \int_0^t b(X_{\tau_s^n}^n)\,\mathrm{d}s \;.$$

(2) Show that there exists a constant $A > 0$ such that

$$\mathbb{E}\Big[(X_t^n - X_{\tau_t^n}^n)^2\Big] \leq \frac{A}{n} \;, \qquad \forall t \in [0,1] \;,\; \forall n \geq 1 \;.$$

(3) Show that for every $n \geq 1$ and every $t \in [0,1]$,

$$\mathbb{E}\big[(X_t - X_t^n)^2\big] \leq 4K^2 \int_0^t \mathbb{E}\Big[(X_s - X_{\tau_s^n}^n)^2\Big]\,\mathrm{d}s \;.$$

Then prove that

$$\mathbb{E}\big[(X_t - X_t^n)^2\big] \leq 8K^2 \int_0^t \mathbb{E}\big[(X_s - X_s^n)^2\big]\,\mathrm{d}s + \frac{32K^2M^2}{n} \;,$$

and deduce that there exists a constant $C_1 > 0$ such that

$$\sup_{t\in[0,1]} \mathbb{E}\big[(X_t - X_t^n)^2\big] \leq \frac{C_1}{n} \;, \qquad \forall n \geq 1 \;.$$

(4) Show that there exists a constant $C_2 > 0$ such that for every bounded continuously differentiable function f,

$$|\mathbb{E}[f(X_1^n) - f(X_1)]| \leq \|f'\|_\infty \frac{C_2}{\sqrt{n}} \;, \qquad \forall n \geq 1 \;.$$

Exercise 9.2. Let $(B_t)_{t\geq 0}$ be a standard $(\mathcal{F}_t)$-Brownian motion.

(1) Show that there exists a continuous adapted process $(X_t)_{t\geq 0}$ such that

$$X_t = 1 + \int_0^t \frac{1}{(1+s)(1+|X_s|)}\,\mathrm{d}B_s - \frac{1}{2}\int_0^t X_s\,\mathrm{d}s, \qquad t \geq 0\ .$$

(2) Show that $(X_t)_{t\geq 0}$ is adapted to the canonical filtration of $(B_t)_{t\geq 0}$.

(3) Define

$$X_t^* = \sup_{s\in[0,t]} |X_s|\ .$$

Show that $\mathbb{E}[(X_1^*)^2] < \infty$. Then, by considering the shifted process $X_{1+t} - X_1$, show that $\mathbb{E}[(X_2^*)^2] < \infty$. Deduce that

$$\mathbb{E}[(X_t^*)^2] < \infty \qquad \text{for every } t \geq 0\ .$$

(4) Let

$$Y_t = \mathrm{e}^{1/(1+t)}(1 + X_t^2), \qquad t \geq 0\ .$$

Show that $(Y_t)_{t\geq 0}$ is a supermartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. Deduce that $\lim_{t\to\infty} Y_t$ exists almost surely and is almost surely finite.

(5) Show that $\lim_{t\to\infty} |X_t|$ exists almost surely.

(6) By considering the finite-variation part of $(Y_t)_{t\geq 0}$, show that

$$\int_0^\infty X_s^2\,\mathrm{d}s < \infty \qquad \text{a.s.}$$

(7) Show that

$$\liminf_{t\to\infty} |X_t| = 0 \qquad \text{a.s.}$$

and deduce that

$$\lim_{t\to\infty} X_t = 0 \qquad \text{a.s.}$$

Solutions

Solution to Exercise 9.1

(1) By construction, on each interval $\left[\frac{k}{n}, \frac{k+1}{n}\right)$, the process X^n satisfies

$$X_t^n = X_{k/n}^n + \sigma(X_{k/n}^n)(B_t - B_{k/n}) + b(X_{k/n}^n)\left(t - \frac{k}{n}\right) .$$

Since $\tau_s^n = \frac{k}{n}$ for $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right)$, summing over the successive intervals gives

$$X_t^n = x + \int_0^t \sigma(X_{\tau_s^n}^n)\,\mathrm{d}B_s + \int_0^t b(X_{\tau_s^n}^n)\,\mathrm{d}s .$$

(2) Fix $t \in [0,1]$ and let k be such that $\frac{k}{n} \le t < \frac{k+1}{n}$. Then

$$X_t^n - X_{\tau_t^n}^n = \sigma(X_{k/n}^n)(B_t - B_{k/n}) + b(X_{k/n}^n)\left(t - \frac{k}{n}\right) .$$

Using $(u+v)^2 \le 2u^2 + 2v^2$, the bound $|\sigma| \vee |b| \le M$, and $\mathbb{E}[(B_t - B_{k/n})^2] = t - \frac{k}{n} \le \frac{1}{n}$, we get

$$\begin{aligned}
\mathbb{E}\Big[(X_t^n - X_{\tau_t^n}^n)^2\Big] &\le 2M^2\,\mathbb{E}\big[(B_t - B_{k/n})^2\big] + 2M^2\left(t - \frac{k}{n}\right)^2 \\
&\le \frac{2M^2}{n} + \frac{2M^2}{n^2} \le \frac{4M^2}{n} .
\end{aligned}$$

So the claim holds with $A = 4M^2$.

(3) Subtracting the equations for $(X_t)_{t\ge 0}$ and X^n gives

$$X_t - X_t^n = \int_0^t \big(\sigma(X_s) - \sigma(X_{\tau_s^n}^n)\big)\,\mathrm{d}B_s + \int_0^t \big(b(X_s) - b(X_{\tau_s^n}^n)\big)\,\mathrm{d}s .$$

Using $(u+v)^2 \le 2u^2 + 2v^2$, Itô's isometry, Cauchy–Schwarz, and $t \le 1$, we obtain

$$\begin{aligned}
\mathbb{E}\big[(X_t - X_t^n)^2\big] &\le 2\,\mathbb{E}\left[\left(\int_0^t (\sigma(X_s) - \sigma(X_{\tau_s^n}^n))\,\mathrm{d}B_s\right)^2\right] \\
&\quad + 2\,\mathbb{E}\left[\left(\int_0^t (b(X_s) - b(X_{\tau_s^n}^n))\,\mathrm{d}s\right)^2\right] \\
&\le 2\int_0^t \mathbb{E}\Big[(\sigma(X_s) - \sigma(X_{\tau_s^n}^n))^2\Big]\,\mathrm{d}s \\
&\quad + 2t\int_0^t \mathbb{E}\Big[(b(X_s) - b(X_{\tau_s^n}^n))^2\Big]\,\mathrm{d}s \\
&\le 4K^2\int_0^t \mathbb{E}\Big[(X_s - X_{\tau_s^n}^n)^2\Big]\,\mathrm{d}s .
\end{aligned}$$

Next,

$$X_s - X_{\tau_s^n}^n = (X_s - X_s^n) + (X_s^n - X_{\tau_s^n}^n) ,$$

so

$$(X_s - X_{\tau_s^n}^n)^2 \le 2(X_s - X_s^n)^2 + 2(X_s^n - X_{\tau_s^n}^n)^2 .$$

Hence

$$\mathbb{E}\big[(X_t - X_t^n)^2\big] \le 8K^2\int_0^t \mathbb{E}\big[(X_s - X_s^n)^2\big]\,\mathrm{d}s + 8K^2\int_0^t \mathbb{E}\Big[(X_s^n - X_{\tau_s^n}^n)^2\Big]\,\mathrm{d}s .$$

By part 2,

$$\mathbb{E}\Big[(X_s^n - X_{\tau_s^n}^n)^2\Big] \le \frac{4M^2}{n} ,$$

therefore

$$\mathbb{E}\big[(X_t - X_t^n)^2\big] \le 8K^2 \int_0^t \mathbb{E}\big[(X_s - X_s^n)^2\big] \,\mathrm{d}s + \frac{32K^2M^2}{n} .$$

Set

$$u_n(t) = \mathbb{E}\big[(X_t - X_t^n)^2\big] .$$

Then

$$u_n(t) \le 8K^2 \int_0^t u_n(s) \,\mathrm{d}s + \frac{32K^2M^2}{n} .$$

By Gronwall's lemma,

$$u_n(t) \le \frac{32K^2M^2}{n} \mathrm{e}^{8K^2 t} \le \frac{C_1}{n}, \qquad t \in [0,1] ,$$

for some constant $C_1 > 0$ independent of n.

(4) Let f be bounded and continuously differentiable. By the mean value theorem,

$$|f(X_1^n) - f(X_1)| \le \|f'\|_\infty |X_1^n - X_1| .$$

Taking expectations and applying Cauchy–Schwarz,

$$|\mathbb{E}[f(X_1^n) - f(X_1)]| \le \|f'\|_\infty \,\mathbb{E}[|X_1^n - X_1|] \le \|f'\|_\infty \,\mathbb{E}[(X_1^n - X_1)^2]^{1/2} .$$

Using part 3 at time $t = 1$,

$$\mathbb{E}[(X_1^n - X_1)^2] \le \frac{C_1}{n} ,$$

hence

$$|\mathbb{E}[f(X_1^n) - f(X_1)]| \le \|f'\|_\infty \frac{\sqrt{C_1}}{\sqrt{n}} .$$

This proves the claim with $C_2 = \sqrt{C_1}$.

Back to Exercise 9.1

Solution to Exercise 9.2

(1) Consider the coefficients

$$\sigma(t,x) = \frac{1}{(1+t)(1+|x|)} \qquad \text{and} \qquad b(t,x) = -\frac{1}{2}x .$$

The drift b is globally Lipschitz in x, and the diffusion coefficient σ is continuous in (t,x) and globally bounded. Moreover, for every fixed t, the map $x \mapsto \sigma(t,x)$ is globally Lipschitz, since

$$\left| \frac{1}{1+|x|} - \frac{1}{1+|y|} \right| \le |x-y| .$$

Therefore the standard existence and uniqueness theorem for SDEs yields a unique strong solution to

$$X_t = 1 + \int_0^t \sigma(s, X_s) \,\mathrm{d}B_s + \int_0^t b(s, X_s) \,\mathrm{d}s ,$$

which is exactly the required equation.

(2) Since the coefficients are non-anticipative and the equation admits a unique strong solution, the solution is measurable with respect to the Brownian path up to time t. Hence $(X_t)_{t\geq 0}$ is adapted to the canonical filtration of $(B_t)_{t\geq 0}$.

(3) We first consider the interval $[0,1]$. From the equation,

$$X_t = 1 + M_t - \frac{1}{2}\int_0^t X_s \,\mathrm{d}s \ ,$$

where

$$M_t = \int_0^t \frac{1}{(1+s)(1+|X_s|)} \,\mathrm{d}B_s \ .$$

Thus

$$X_t^* \leq 1 + \sup_{s\in[0,t]} |M_s| + \frac{1}{2}\int_0^t X_u^* \,\mathrm{d}u \ .$$

By Gronwall's lemma,

$$X_1^* \leq C\left(1 + \sup_{s\in[0,1]} |M_s|\right)$$

for some deterministic constant C. Therefore

$$\mathbb{E}[(X_1^*)^2] \leq C\left(1 + \mathbb{E}\left[\sup_{s\in[0,1]} |M_s|^2\right]\right) \ .$$

By Doob's inequality,

$$\mathbb{E}\left[\sup_{s\in[0,1]} |M_s|^2\right] \leq 4\,\mathbb{E}[\langle M\rangle_1] = 4\,\mathbb{E}\left[\int_0^1 \frac{1}{(1+s)^2(1+|X_s|)^2}\,\mathrm{d}s\right] \leq 4\int_0^1 \frac{1}{(1+s)^2}\,\mathrm{d}s < \infty \ .$$

Hence $\mathbb{E}[(X_1^*)^2] < \infty$.

Now consider the shifted process

$$\widetilde{X}_t = X_{1+t}, \qquad t \geq 0 \ .$$

Then, for $t \in [0,1]$,

$$\widetilde{X}_t - X_1 = \int_1^{1+t} \frac{1}{(1+s)(1+|X_s|)}\,\mathrm{d}B_s - \frac{1}{2}\int_1^{1+t} X_s \,\mathrm{d}s \ .$$

Exactly the same argument on $[1,2]$ shows that

$$\mathbb{E}\left[\sup_{t\in[1,2]} |X_t - X_1|^2\right] < \infty \ .$$

Since $\mathbb{E}[X_1^2] < \infty$, it follows that $\mathbb{E}[(X_2^*)^2] < \infty$.

Repeating the argument on each interval $[k, k+1]$ yields

$$\mathbb{E}[(X_t^*)^2] < \infty \qquad \text{for every } t \geq 0 \ .$$

(4) Apply Itô's formula to X_t^2:

$$\mathrm{d}(X_t^2) = 2X_t\,\mathrm{d}X_t + \mathrm{d}\langle X\rangle_t \ .$$

Since

$$\mathrm{d}X_t = \frac{1}{(1+t)(1+|X_t|)}\,\mathrm{d}B_t - \frac{1}{2}X_t\,\mathrm{d}t$$

and

$$\mathrm{d}\langle X\rangle_t = \frac{1}{(1+t)^2(1+|X_t|)^2}\,\mathrm{d}t\ ,$$

we get

$$\mathrm{d}(X_t^2) = \frac{2X_t}{(1+t)(1+|X_t|)}\,\mathrm{d}B_t - X_t^2\,\mathrm{d}t + \frac{1}{(1+t)^2(1+|X_t|)^2}\,\mathrm{d}t\ .$$

Now set

$$Y_t = \mathrm{e}^{1/(1+t)}(1+X_t^2)\ .$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{1/(1+t)} = -\frac{\mathrm{e}^{1/(1+t)}}{(1+t)^2}\ ,$$

Itô's formula gives

$$\begin{aligned}\mathrm{d}Y_t &= \mathrm{e}^{1/(1+t)}\,\mathrm{d}(X_t^2) - \frac{\mathrm{e}^{1/(1+t)}}{(1+t)^2}(1+X_t^2)\,\mathrm{d}t\\ &= \frac{2\mathrm{e}^{1/(1+t)}X_t}{(1+t)(1+|X_t|)}\,\mathrm{d}B_t\\ &\quad + \mathrm{e}^{1/(1+t)}\left[-X_t^2 + \frac{1}{(1+t)^2(1+|X_t|)^2} - \frac{1+X_t^2}{(1+t)^2}\right]\mathrm{d}t\ .\end{aligned}$$

Since

$$\frac{1}{(1+|x|)^2} \le 1\ ,$$

the drift satisfies

$$-X_t^2 + \frac{1}{(1+t)^2(1+|X_t|)^2} - \frac{1+X_t^2}{(1+t)^2} \le -X_t^2 - \frac{X_t^2}{(1+t)^2} \le 0\ .$$

Hence $(Y_t)_{t\ge 0}$ is a supermartingale.

Since $Y_t \ge 0$, the supermartingale convergence theorem implies that

$$\lim_{t\to\infty} Y_t$$

exists almost surely and is finite almost surely.

(5) Since

$$\mathrm{e}^{1/(1+t)} \longrightarrow 1 \qquad \text{as } t\to\infty\ ,$$

and

$$Y_t = \mathrm{e}^{1/(1+t)}(1+X_t^2)\ ,$$

the almost sure convergence of Y_t implies the almost sure convergence of $1+X_t^2$, hence of X_t^2. Therefore

$$\lim_{t\to\infty} |X_t|$$

exists almost surely.

(6) From the computation in part 4, we can write

$$Y_t = Y_0 + N_t - A_t\ ,$$

where N is a local martingale and

$$A_t = \int_0^t \mathrm{e}^{1/(1+s)}\left[X_s^2 + \frac{1+X_s^2}{(1+s)^2} - \frac{1}{(1+s)^2(1+|X_s|)^2}\right]\mathrm{d}s$$

is increasing.

Since Y_t converges almost surely and is nonnegative, its finite-variation part must remain finite as $t \to \infty$. In particular,

$$\int_0^\infty e^{1/(1+s)} X_s^2 \, ds < \infty \qquad \text{a.s.}$$

Because $e^{1/(1+s)} \geq 1$, we deduce that

$$\int_0^\infty X_s^2 \, ds < \infty \qquad \text{a.s.}$$

(7) Suppose that on an event of positive probability,

$$\liminf_{t \to \infty} |X_t| = \ell > 0 \, .$$

Then there exist $\varepsilon > 0$ and $T < \infty$ such that on this event,

$$|X_t| \geq \varepsilon \qquad \text{for all } t \geq T \, .$$

This would imply

$$\int_T^\infty X_s^2 \, ds \geq \int_T^\infty \varepsilon^2 \, ds = \infty \, ,$$

contradicting part 6. Therefore

$$\liminf_{t \to \infty} |X_t| = 0 \qquad \text{a.s.}$$

By part 5, the limit

$$\lim_{t \to \infty} |X_t|$$

exists almost surely. Since its liminf is 0, the limit itself must be 0. Hence

$$\lim_{t \to \infty} |X_t| = 0 \qquad \text{a.s.}$$

and therefore

$$\lim_{t \to \infty} X_t = 0 \qquad \text{a.s.}$$

Back to Exercise 9.2