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Stochastic calculus for machine learning, Master MVA

3 Brownian motion: Markov property and quadratic variation

If not specified, $(B_t)_{t\geq 0}$ denotes a standard Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration. In addition, $(S_t)_{t\geq 0}$ is the process defined as

$$S_t = \sup_{s\in[0,t]} B_s \,, t \geq 0 \,. \tag{14}$$

Exercise 3.1. Let $\tau_a = \inf\{t \geq 0 \,:\, B_t = a\}$ for any $a \in \mathbb{R}$.

(1) Show that $\tau_a < \infty$ almost surely.

(2) Show that for any $c > 0$, the process $(\tau_a)_{a\in\mathbb{R}_+}$ has the same distribution as $(c^{-2}\tau_{ca})_{a\in\mathbb{R}_+}$.

Exercise 3.2 (Reflection principle)**.** Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and define

$$S_t = \sup_{s\in[0,t]} B_s \,, \qquad t > 0 \,.$$

In addition, define $\tau_a = \inf\{t \geq 0 \,:\, B_t = a\}$ for any $a \in \mathbb{R}$.

(1) Show that $\tau_a < \infty$ almost surely.

Let $a \geq 0$ and $b \leq a$ and $t > 0$.

(2) Show that

$$\mathbb{P}(S_t \geq a \,,\, B_t \leq b) = \mathbb{P}(\tau_a \leq t \,,\, \tilde{B}_{t-\tau_a} \leq b - a) \,, \tag{15}$$

where $\tilde{B}_t = B_{t+\tau_a} - B_{\tau_a}$.

(3) Show that $(\tilde{B}_t)_{t\geq 0}$ is BM independent of τ_a.

(4) Deduce that $\mathbb{P}(\tau_a \leq t \,,\, \tilde{B}_{t-\tau_a} \leq b - a) = \mathbb{P}(\tau_a \leq t \,,\, -\tilde{B}_{t-\tau_a} \leq b - a)$.

(5) Deduce that

$$\mathbb{P}(S_t \geq a,\ B_t \leq b) = \mathbb{P}(B_t \geq 2a - b) \,. \tag{16}$$

(6) Deduce that $\mathbb{P}(S_t \geq a) = 2\mathbb{P}(B_t \geq a)$ and therefore the random variables S_t and $|B_t|$ have the same distribution.

(7) Deduce that the random vector (S_t, B_t) admits a joint density f_{S_t,B_t} with respect to Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}$, given by

$$f_{S_t,B_t}(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) \mathbb{1}_{\{a>0,\ b<a\}} \,.$$

(8) Compute $\mathbb{P}(S_t \geq a)$ and show that the density with respect to the Lebesgue measure of τ_a is given by

$$f_{\tau_a}(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) \mathbb{1}_{\mathbb{R}_+^*}(t) \,. \tag{17}$$

Exercise 3.3. Define

$$\tau_{\geq 1,0} := \inf\{t \geq 1 : B_t = 0\} \,, \qquad \sigma_{\sup,1} := \sup\{t \leq 1 : B_t = 0\} \,.$$

(1) Is the random variable $\tau_{\geq 1,0}$ a $(\mathcal{F}_t)_{t\geq 0}$-stopping time?

(2) Compute the law of $\tau_{\geq 1,0}$ and the law of $\sigma_{\sup,1}$ (hint: use the Markov property and the known formula for the law of τ_x, $x \in \mathbb{R}$).

(3) Is the random variable $\sigma_{\sup,1}$ a $(\mathcal{F}_t)_{t\geq 0}$-stopping time?

Exercise 3.4. Let

$$\tau_1 := \inf\{t > 0 : B_t = 1\} , \qquad \tau := \inf\{t \geq \tau_1 : B_t = 0\} .$$

(1) Is τ a stopping time?

(2) Compute the law of τ.

Exercise 3.5. (1) Analyze the convergence in distribution as $t \to +\infty$ of the process $(X_t)_{t\geq 0}$ defined as

$$X_t = \frac{\log(1 + B_t^2)}{\log t} .$$

(2) What can be said about its convergence in probability?

(3) And its convergence almost sure?

Exercise 3.6. (1) Let $0 < a < b < c < d$. Show that almost surely,

$$\sup_{t\in[a,b]} B_t \neq \sup_{t\in[c,d]} B_t .$$

(2) Deduce that almost surely every local maximum of $(B_t)_{t\geq 0}$ is a strict local maximum.

Exercise 3.7. (1) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and define for $\lambda > 0$, $\psi(\lambda) = \lambda^{-1} \log \int_0^1 e^{\lambda f(t)} dt$. Show that

$$\lim_{\lambda\to+\infty} \psi(\lambda) = \sup_{[0,1]} f . \tag{18}$$

(2) Define $Z_t = (\int_0^t e^{B_s} ds)^{1/\sqrt{t}}$ for any $t \geq 0$. Using the scaling property of Brownian motion, deduce that $(Z_t)_{t\geq 0}$ converges in distribution to $e^{|B_1|}$.

Exercise 3.8. Let $a > 0$ and define $\tau_a := \inf\{t > 0 : B_t = a\}$. Recall that

$$\mathbb{P}(\tau_a \leq t) \leq \exp\left(-\frac{a^2}{2t}\right) , \qquad t > 0 .$$

Show that if G is a standard Gaussian random variable, then

$$\mathbb{P}(G \geq x) \leq \frac{1}{2} e^{-x^2/2} , \qquad x > 0 .$$

Exercise 3.9. Show that $S_2 - S_1$ has the same distribution as $\max\{|G| - |\tilde{G}|, 0\}$, where G and $\tilde{G}$ are independent standard Gaussian random variables.

Exercise 3.10. Show, without using time inversion, but using the law of large numbers and the reflection principle, that $B_t/t \to 0$ almost surely as $t \to \infty$.

Exercise 3.11. Show that, almost surely $\int_0^\infty \sin^2(B_t)\, dt = \infty$.

Exercise 3.12. (1) Show that there exists $c > 0$ such that for all $t \geq 1$,

$$\mathbb{P}\left(\sup_{s\in[0,t]} |B_s| \leq 2\right) \geq e^{-ct} .$$

(2) Show that there exists $c > 0$ such that for all $\varepsilon \in (0, 1]$,

$$\mathbb{P}\left(\sup_{s\in[0,1]} |B_s| \leq \varepsilon \right) \geq \mathrm{e}^{-c/\varepsilon^2} .$$

(3) Show that for all $t > 0$ and all $x > 0$,

$$\mathbb{P}\left(\sup_{s\in[0,t]} |B_s| \geq x \right) > 0 .$$

Exercise 3.13 (Law of the Iterated Logarithm)**.** Define for any $t > 0, h(t) := \sqrt{2t \log \log t}$.

(1) Let $\varepsilon > 0$ and define $t_n := (1+\varepsilon)^n$. Show that $\sum_n \mathbb{P}\big(S_{t_{n+1}} \geq (1 + \varepsilon)h(t_n)\big) < +\infty$ and deduce that

$$\limsup_{t\to\infty} \frac{S_t}{h(t)} \leq 1 \quad \text{a.s.}$$

(2) Show that almost surely $\limsup_{t\to\infty} \sup_{s\in[0,t]} |B_s|/h(t) \leq 1$

(3) Let $\theta > 1$ and define $s_n := \theta^n$. Show that for every $\alpha \in \big(0, \sqrt{1 - \theta^{-1}}\big]$,

$$\sum_n \mathbb{P}\big(B_{s_n} - B_{s_{n-1}} \geq \alpha h(s_n)\big) = +\infty .$$

(4) Deduce that almost surely $\limsup_{t\to\infty} \frac{B_t}{h(t)} \geq \alpha$

(5) Show that almost surely $\limsup_{t\to\infty} \frac{B_t}{h(t)} = 1$.

(6) Let $X^1_t = |B_t|, X^2_t = S_t$ and $X^3_t = \sup_{s\in[0,t]} |B_s|$. What can be said about

$$\limsup_{t\to\infty} \frac{X^i_t}{h(t)}, \qquad i = 1, 2, 3$$

(7) What can be said about $\liminf_{t\to\infty} \frac{B_t}{h(t)}$ and $\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t \log\log(1/t)}}$?