A short course in Markov chains

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Chapter 1

Spectral theory

Time schedule (**Note** 1): Last session.

1.1 Markov operator

We define $L_2(\pi)$ as the set of measurable functions f on X such that $\pi(f^2) < \infty$. The space $(L_2(\pi), \|\cdot\|)$, where the norm is induced by the inner product on $L_2(\pi)$, $\langle f, g\rangle = \int \pi(\mathrm{d}x) f(x) g(x)$, is a Hilbert space.

Bounded linear operator

LEMMA 1.1 . Let P be a Markov kernel on $(\mathsf{X}, \mathscr{X})$ admitting π as an invariant probability measure. Then

$$P : f \mapsto Pf$$

is a bounded linear operator on $L_2(\pi)$. Moreover, $|||P||| = 1$.

Hence, P induces a bounded linear operator on $L_2^0(\pi)$ and for notational convenience, in what follows, we use the same notation P, seen either as a Markov kernel or as an operator on $L_2^0(\pi)$.

PROOF. For any $f \in L_2(\pi)$, we have $\pi\left[(Pf)^2\right] \leqslant \pi(P[f^2]) = \pi(f^2)$, which shows that P maps $L_2(\pi)$ into itself. The operator is clearly linear. The previous inequality can also be written as $\|Pf\|^2 \leqslant \|f\|^2$, and therefore $|||P||| \leqslant 1$. This shows that P is a bounded linear operator on $L_2(\pi)$. Since $P1 = 1$, we obtain $\|P1\| = \|1\|$, and hence $|||P||| = 1$. ■

Most of the time, we work with real-valued functions. When studying the spectrum and the resolvent set, we implicitly consider the complexification of $L_2(\pi)$, in which case, the inner-product will be $\langle f, g\rangle = \int \pi(\mathrm{d}x) \bar{f}(x) g(x)$. Moreover, in these lecture notes, most of the results, although stated for the Markov operator P actually hold more generally for any bounded linear operator. We focus on Markov operators only to avoid unnecessary generality. We denote by $\mathsf{BL}_2(\pi)$ the set of bounded linear operators on $L_2(\pi)$. We define:

- $\mathrm{Spec}(P) = \{\lambda \in \mathbb{C} : \lambda I - P \text{ is not invertible}\}$, the spectrum of P.
- $\mathrm{Spec}_p(P) = \{\lambda \in \mathbb{C} : \mathrm{Ker}[\lambda I - P] \neq \{0\}\}$, the point spectrum of P.

Clearly, $\mathrm{Spec}_p(P) \subset \mathrm{Spec}(P)$. Moreover, if $S \in \mathsf{BL}_2(\pi)$ with $|||S||| < 1$, then the series $\sum_k S^k$ is normally convergent and can be shown to be the inverse of $I - S$. It follows that for any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$, $\lambda I - P = \lambda(I - P/\lambda)$ is invertible, and therefore $\mathrm{Spec}(P) \subset \bar{\mathrm{B}}(0,1)$.

The resolvent set of P is defined by $\mathrm{Res}(P) = \mathrm{Spec}(P)^c$. It is an open set. Indeed, if S is invertible, then writing $T = S(S^{-1}(T-S)+I)$ and taking T sufficiently close to S, we see that T is invertible with inverse $(I+S^{-1}(T-S))^{-1}S^{-1}$.

By definition, an eigenvalue λ of P is an element of the point spectrum; its multiplicity is $\mathrm{Dim}(\lambda I - P)$.

Note that 1 is an eigenvalue of P with multiplicity 1. Indeed, assume that there exists a function $f \in L_2(\pi)$ satisfying $Pf = f$. Then $\pi(f^2) = \pi([Pf]^2) \leqslant \pi(P[f^2]) = \pi(f^2)$ which implies that we have equality in the Cauchy Schwarz inequality: for $\mathbb{P}_\pi$-almost all $x \in \mathsf{X}$, $(Pf(x))^2 = P[f^2](x)$ and hence, f is $\pi - a.s.$ constant. (Concerning this argument, see also the comments in the appendix).

We thus obtain the orthogonal decomposition $L_2(\pi) = \mathrm{Span}(1) \overset{\perp}{\oplus} L_2^0(\pi)$, where $L_2^0(\pi) = \{f \in L_2(\pi) : \pi(f) = 0\}$ is a closed subspace, invariant under P.

We are therefore interested in the asymptotic behaviour of

$$\sup_{f \in L_2^0(\pi)} \|P^n f\| = |||P^n|||_{L_2^0(\pi)} = \sup_{h \in L_2(\pi)} \|P^n h - \pi(h)\|.$$

For convenience, we set $H = L_2(\pi)$ and $H_0 = L_2^0(\pi)$.

THEOREM 1.2 . Defining the spectral radius by $\mathrm{Spec.Rad.}(P|_{H_0}) = \{|\lambda| : \lambda \in \mathrm{Spec}(P|_{H_0})\}$, we have

$$\mathrm{Spec.Rad.}(P|_{H_0}) = \lim_n |||P^n|||_{H_0}^{1/n}.$$

PROOF. Let A denote the left-hand side and B the right-hand side. The existence of the limit appearing in the expression of B follows from Fekete's lemma, since by setting $a_n = |||P^n|||$, one has $\log a_{p+q} \leqslant \log a_p + \log a_q$, which implies that $\lim \log a_n/n$ converges, its limit being equal to $\inf_n \log a_n/n$, a limit which may in fact be $-\infty$.

Let us now show that $A \leqslant B$, which is the easier direction. If we choose $\lambda \in \mathbb{C}$ such that $|\lambda| > B$, then the series $\sum_n (P/\lambda)^n$ converges normally and is the inverse of $I - P/\lambda$, which shows that $\lambda I - P$ is invertible. Hence λ belongs to the resolvent set. Therefore $A \leqslant |\lambda|$. Finally, $A \leqslant B$.

We now show $B \leqslant A$. Let us take $\lambda \in \mathbb{C}$ such that $|\lambda| > A$. Then one may define $\phi(z) = (I - zP)^{-1}$ for all $|z| < \lambda^{-1}$. We now prove the Cauchy integral formula. Readers may safely skip this proof on a first reading; it is included for completeness and for its elegance and usefulness. For any $r < |\lambda|^{-1}$, and any $z_0 \in \mathbb{C}$ with $|z_0| < r$, define $g(\beta) = \int_0^{2\pi} \frac{\phi(\beta r e^{i\theta} + (1-\beta)z_0)}{re^{i\theta} - z_0} re^{i\theta}\, \mathrm{d}\theta$. Since

$$g'(\beta) = \int_0^{2\pi} \phi'(\beta r e^{i\theta} + (1-\beta)z_0)\, re^{i\theta}\, \mathrm{d}\theta = \left[\frac{\phi(\beta r e^{i\theta} + (1-\beta)z_0)}{i\beta}\right]_0^{2\pi} = 0,$$

we deduce that g is constant and in particular that $g(0) = g(1)$, which can be rewritten as

$$\phi(z_0)\int_0^{2\pi} \frac{1}{1-(z_0/r)e^{-i\theta}}\, \mathrm{d}\theta = \int_0^{2\pi} \frac{\phi(re^{i\theta})}{1-(z_0/r)e^{-i\theta}}\, \mathrm{d}\theta.$$

Expanding inside the integral, $(1-(z_0/r)e^{-i\theta})^{-1} = \sum_n (z_0/r)^n e^{-i\theta n}$, and interchanging (legitimately) the series and the integral, we obtain, for z_0 in a neighborhood of 0,

$$\phi(z_0) = \sum_{n=0}^{\infty} z_0^n \frac{1}{2\pi} \frac{\int_0^{2\pi} \phi(re^{i\theta}) e^{-in\theta}\, \mathrm{d}\theta}{r^n} = (I - z_0 P)^{-1} = \sum_{n=0}^{\infty} z_0^n P^n.$$

At this point, we may equate the Taylor expansions, which yields

$$P^n = \frac{1}{2\pi} \frac{\int_0^{2\pi} \phi(re^{i\theta}) e^{-in\theta}\, \mathrm{d}\theta}{r^n}$$

for all $n \in \mathbb{N}$. Since ϕ is continuous and therefore bounded on any compact set, there exists a constant C such that $|||P^n||| \leqslant C/r^n$, and hence $\limsup_n |||P^n|||^{1/n} \leqslant 1/r$. As this holds for any $r < |\lambda|^{-1}$, we obtain $B \leqslant |\lambda|$. Finally, $B \leqslant A$, and the proof is complete. ∎

Comment on the proof. To be precise, a careful reading of this proof shows that the mapping $\phi : \mathbb{C} \to L_2^0(\pi)$ should be complex differentiable, that is, holomorphic on the ball $B(0,|\lambda|^{-1})$. This property can be verified directly on the resolvent set of P. Indeed, writing

$$I-(z+h)P = (I-zP)\left[(I-zP)^{-1}(-hP)+I\right]$$

we see that, for h sufficiently small, the inverse of $I-(z+h)P$ is given by

$$(I-(z+h)P)^{-1} = \left[(I-zP)^{-1}(-hP)+I\right]^{-1}(I-zP)^{-1} = \sum_{k=0}^{\infty}(-h)^k\left[(I-zP)^{-1}P\right]^k(I-zP)^{-1}.$$

This expansion implies $\phi(z+h) = \phi(z) - h(I-zP)^{-1}P(I-zP)^{-1} + o(|h|)$, showing that ϕ is indeed holomorphic at z.

1.2 Reversibility and self-adjointness.

We now assume that $\boxed{P \text{ is } \pi\text{-reversible}}$, that is $\pi(\mathrm{d}x)P(x,\mathrm{d}y) = \pi(\mathrm{d}y)P(y,\mathrm{d}x)$. Then, obviously P is self-adjoint, i.e., $\langle Pf, g\rangle = \langle f, Pg\rangle$. Note that since P is self-adjoint, $\langle Pf, f\rangle \in \mathbb{R}$.

Theorem 1.3 . If P is reversible, then

$$\begin{aligned}|||P|||_{L_2^0(\pi)} &= \sup_{\|f\|\leqslant 1, f\in L_2^0(\pi)} \sqrt{\langle Pf, Pf\rangle} = \sup_{\|f\|\leqslant 1, f\in L_2^0(\pi)} |\langle Pf, f\rangle| \\ &= \lim_{n\to\infty} |||P^n|||_{L_2^0(\pi)}^{1/n} = \sup\{|\lambda| : \lambda \in \mathrm{Spec}(P|_{L_2^0(\pi)})\}.\end{aligned}$$

Proof. Let us denote the previous equalities by $A = B = C = D = E$. By definition of the triple norm, we clearly have $A = B$. Moreover, $C \leqslant B$ follows from the Cauchy–Schwarz inequality. To show that $B \leqslant C$, write $\langle Pf, \underbrace{Pf/\|Pf\|}_{g}\rangle$ for f of norm 1 and express this quantity in terms of $\langle P(f\pm g), f\pm g\rangle$ using the parallelogram identity (and the fact that P is self-adjoint). We obtain, noting that $\langle Pf, g\rangle = \|Pf\| \in \mathbb{R}$,

$$\|Pf\| = |\langle Pf, g\rangle| = \left|\frac{1}{4}\left[\langle P(f+g), f+g\rangle - \langle P(f-g), f-g\rangle\right]\right| \leqslant \frac{C}{4}\left(\|f+g\|^2 + \|f-g\|^2\right) \leqslant C.$$

Hence $B = C$. Thus $A = B = C$ and $D = E$ (by Theorem 1.2).

Finally, it remains to show that $A = D$. Using the identity $\langle Pf, Pf\rangle = \langle P^2 f, f\rangle$ in the equalities $A = B = C$, we obtain $|||P|||^2 = |||P^2|||$. By induction, this yields $|||P|||^{2^k} = |||P^{2^k}|||$. Therefore, $|||P||| = |||P^{2^k}|||^{1/2^k}$, and letting k tend to infinity, we conclude that $A = D$.

■

The following theorem is stated for the Markov operator P, but we emphasize that it applies more generally to any self-adjoint bounded operator.

Theorem 1.4 . If P is self-adjoint, then its eigenvalues are real and

$$\mathrm{Spec}(P|_{L_2^0(\pi)}) \subset [m, M],$$

where

$$m = \inf_{f\in L_2^0(\pi), \|f\|\leqslant 1} \langle Pf, f\rangle, \qquad M = \sup_{f\in L_2^0(\pi), \|f\|\leqslant 1} \langle Pf, f\rangle,$$

and both bounds m and M belong to the spectrum of P.

PROOF. Let $z \notin [m,M]$ and let us show that it belongs to the resolvent set. Let $f \in H_0$. Choose α such that $\langle(\alpha I - P)f, f\rangle = 0$. Then

$$\|(zI-P)f\|^2 = \|(\alpha I - P)f\|^2 + |z-\alpha|^2\|f\|^2 \geqslant |z-\alpha|^2\|f\|^2 \geqslant \Delta\|f\|^2,$$

where we have set $\Delta = d(z,[m,M]) > 0$. This simple inequality shows that z belongs to the resolvent set. Indeed, it successively implies that $\mathrm{Ker}(zI-P) = \{0\}$, that $\mathrm{Ran}(zI-P)$ is closed, and that if $g \in \mathrm{Ran}(zI-P)^\perp$ then $(\bar{z}I-P)g = 0$. Applying the above inequality with $(\bar{z},g)$ instead of (z,f), we obtain that $g=0$. Hence $zI-P$ is invertible and, moreover, its inverse is bounded (again by the same inequality). Therefore, z belongs to the resolvent set. This proves the first part of the theorem.

Finally, suppose that $M = |||P|||$. Choose f_n of norm 1 such that $\langle Pf_n, f_n\rangle \to M$. Then

$$\begin{aligned}\|(MI-P)f_n\|^2 &= M^2 + \|Pf_n\|^2 - 2M\langle f_n, Pf_n\rangle \\ &\leqslant 2M^2 - 2M\langle f_n, Pf_n\rangle \to 2M^2 - 2M^2 = 0.\end{aligned}$$

Thus $MI-P$ is not invertible (otherwise, $1 = \|f_n\|^2 \leqslant |||(MI-P)^{-1}|||\,\|(MI-P)f_n\| \to 0$). Therefore $M \in \mathrm{Spec}(P)|_{H_0}$.

So far, the proof was written with P but this also holds for any self-adjoint bounded operator. This remark allows to replace P by $Q = MI - P$, we obtain

$$\sup_{f\in L_2^0(\pi),\ \|f\|\leqslant 1} \langle Qf, f\rangle = M - m,$$

$$\inf_{f\in L_2^0(\pi),\ \|f\|\leqslant 1} \langle Qf, f\rangle = 0.$$

It follows that $|||Q||| = M - m$ and consequently (by the previous argument applied with P replaced by $MI-P$) that $M - m \in \mathrm{Spec}(MI-P)|_{H_0}$. This means that

$$(M-m)I - (MI-P) = -(mI-P)$$

is not invertible. Hence we have shown that $m \in \mathrm{Spec}(P)$. If we now suppose that $-m = |||P|||$, we apply the same reasoning by replacing P with $-P$.

■

A careful inspection of the proof actually shows that

- $\mathrm{Spec}(P|_{L_2^0(\pi)}) \subset \overline{\{\langle f, Pf\rangle \,:\, f \in L_2^0(\pi), \|f\| \leqslant 1\}}$.

1.2.1 Spectral measure

THEOREM 1.5 . If P is self-adjoint, then for any $f \in L_2^0(\pi)$ there exists a finite (nonnegative) measure μ_f supported on $\mathrm{Spec}(P|_{L_2^0(\pi)}) \subset [-1,1]$ such that, for all $n \in \mathbb{N}$,

$$\langle f, P^n f\rangle = \int_{-1}^{1} x^n \mu_f(\mathrm{d}x).$$

Taking $n = 0$ yields $\mu_f([-1,1]) = \pi(f^2) = \mathrm{Var}_\pi(f)$.

The proof of the theorem is omitted; we only sketch the main ideas. We first give a precise meaning to the map $\phi \mapsto \langle f, \phi(P)f\rangle$: it is initially defined for polynomials and then extended to any continuous function ϕ on $\mathrm{Spec}(P|L_2^0(\pi))$ by density of the polynomials, using the Stone–Weierstrass theorem. This construction yields a nonnegative continuous linear functional on the space of continous functions on the compact set $\mathrm{Spec}(P|L_2^0(\pi))$, equipped with the supremum norm. The existence of the spectral measure then follows from the Riesz representation theorem.

This theorem allows one to replace P^n by the scalar x^n, which greatly simplifies many arguments and is justified by the spectral theorem. Note that μ_f may charge the points $\{1\}$ or $\{-1\}$.

We define

- $\mathrm{Abs.Spec.Gap}(P) = 1 - \sup\{|\lambda| : \lambda \in \mathrm{Spec}(P|_{H_0})\}$,
- $\mathrm{SpecGap}(P) = 1 - \sup\{\lambda : \lambda \in \mathrm{Spec}(P|_{H_0})\}$.

Moreover, we have the following result.

Proposition 1.6 . Let P be a reversible Markov kernel.

$$\begin{aligned}\mathrm{SpecGap}(P) &= \inf_{f\in H_0, \|f\|\leqslant 1} \langle f,f\rangle - \langle Pf,f\rangle \\ &= \inf_{f\in H_0, \|f\|\leqslant 1} \langle (I-P)f,f\rangle \\ &= \inf_{f\in H_0, \|f\|\leqslant 1} \frac{1}{2}\int \pi(\mathrm{d}x)P(x,\mathrm{d}y)(f(y)-f(x))^2.\end{aligned}$$

Some comments. To see the first equality, recall that $I-P$ being reversible, applying Theorem 1.4, with P replaced by $I-P$,

$$\mathrm{SpecGap}(P) = \inf\{\lambda\in\mathbb{C} \;:\; \lambda\in\mathrm{Spec}(I-P)|_{H_0}\} = \inf\left\{\langle f,(I-P)f\rangle \;:\; f\in L_2^0(\pi), \|f\|\leqslant 1\right\}$$

The second equality is immediate. The third follows by expanding $\frac{1}{2}\int\pi(\mathrm{d}x)P(x,\mathrm{d}y)(f(y)-f(x))^2$ and using that P is π-invariant. The standard notation for the Dirichlet form is $\mathscr{E}(f,g) = \langle f,(I-P)g\rangle$. We thus have two equivalent expressions for the Dirichlet form $\mathscr{E}(f,f)$:

$$\mathscr{E}(f,f) = \langle f,(I-P)f\rangle = \frac{1}{2}\int \pi(\mathrm{d}x)P(x,\mathrm{d}y)(f(y)-f(x))^2.$$

If P is positive, that is, $\langle f,Pf\rangle \geqslant 0$ for all $f\in L_2^0(\pi)$, then $\mathrm{Spec}(P|_{H_0})\subset[0,1]$ and the spectral gap coincides with the absolute spectral gap, which allows to combine Theorem 1.3 and Proposition 1.6.

1.3 Comparison of asymptotic behavior for two Markov kernels

As a byproduct of the different expressions of the spectral gap in Proposition 1.6, we have

Corollary 1.7 . Let P and Q be reversible kernels and assume that $P\succeq Q$ in the sense of covariance ordering, that is, $\langle Pf,f\rangle \leqslant \langle Qf,f\rangle$ for all $f\in H_0$. Then:

- $\mathrm{SpecGap}(P) \geqslant \mathrm{SpecGap}(Q)$.
- $\lim_{n\to\infty}\mathrm{Var}_P\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(X_k)\right] \leqslant \lim_{n\to\infty}\mathrm{Var}_Q\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(X_k)\right]$, where both chains are started from the stationnary distribution π.

The first bullet follows immediately from covariance ordering: $\langle Pf,f\rangle \leqslant \langle Qf,f\rangle$ is equivalent to $\langle (I-Q)f,f\rangle \leqslant \langle (I-P)f,f\rangle$. If the spectral gaps are positive, then P converges geometrically to π at a faster rate than Q.

The second is more delicate and corresponds to the proof of Tierney (1998). It shows that the Monte Carlo estimator of $\pi(f)$ has a smaller asymptotic variance when using P rather than Q.

How can covariance ordering be verified? In many cases, it suffices to show that for all $(x,A) \in \mathsf{X} \times \mathscr{X}$,

$$P(x, A \setminus \{x\}) \geqslant Q(x, A \setminus \{x\}),$$

which is known as Peskun ordering. Indeed,

$$\frac{1}{2}\int \pi(\mathrm{d}x)P(x,\mathrm{d}y)(f(y)-f(x))^2 \geqslant \frac{1}{2}\int \pi(\mathrm{d}x)Q(x,\mathrm{d}y)(f(y)-f(x))^2,$$

which implies $\langle (I-P)f, f\rangle \geqslant \langle (I-Q)f, f\rangle$ and hence $P \succeq Q$.

The following exercise allows to prove the second bullet in Corollary 1.7.

Exercise 1 . Let P be a reversible Markov kernel and let $f \in H_0$. Define $A_n = \mathrm{Var}_P\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(X_k)\right]$.

1. Show that
$$A_n = \langle f, f\rangle + 2\sum_{\ell=1}^{n-1}\frac{n-\ell}{n}\langle f, P^\ell f\rangle$$
2. Deduce that there exists a finite non-negative measure μ_f on $[-1,1]$ such that $A_n = \int_{[-1,1]} w_n(x)\mu_f(\mathrm{d}x)$ where
$$w_n(x) = \frac{1+x}{1-x} - \frac{2x}{(1-x)^2}\frac{1-x^n}{n}$$
3. By splitting the integral on $[-1,0]$ and $(0,1]$ show that $\lim_{n\to\infty} A_n$ exists and is equal to $-\langle f, f\rangle + 2\int_{-1}^{1}\frac{1}{1-x}\mu_f(\mathrm{d}x)$.

We now consider two π-reversible kernels P_0, P_1 such that $P_0 \succeq P_1$ according to the covariance ordering. Define $P_\alpha = (1-\alpha)P_0 + \alpha P_1$ for $\alpha \in (0,1]$ and for any $\lambda \in (0,1)$, write $H_\lambda(\alpha) = (I - \lambda P_\alpha)^{-1}$.

4. Show that H'_λ (the right derivative of H_λ) is equal to
$$H'_\lambda(\alpha) = \lambda(I-\lambda P_\alpha)^{-1}(P_1 - P_0)(I - \lambda P_\alpha)^{-1}.$$
5. Using that P_0, P_1 are π-reversible, show that $\langle f, H'_\lambda(\alpha) f\rangle \geqslant 0$.
6. Deduce $\langle f, H_\lambda(0) f\rangle \leqslant \langle f, H_\lambda(1) f\rangle$.
7. Letting $\lambda \to 1$, deduce that
$$\lim_{n\to\infty}\mathrm{Var}_{P_0}\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(X_k)\right] \leqslant \lim_{n\to\infty}\mathrm{Var}_{P_1}\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(X_k)\right].$$

A Appendix

The following lemma can be used to show that 1 has multiplicity 1. The proof, however, is more involved (although also more general since it applies to $f \in L_1(\pi)$ rather than $f \in L_2(\pi)$) than the nice elementary argument presented in the Lecture Notes (originally due to an MDA student in 2025). We include the proof for $f \in L_1(\pi)$ below for completeness.

Lemma .8 . If P admits a unique invariant probability measure π, then any harmonic function $f \in L_1(\pi)$ is $\mathbb{P}_\pi - a.s.$ constant.

PROOF. From $Pf = f$, we deduce that $\{f(X_n)\ :\ n \in \mathbb{N}\}$ is a martingale and that $\sup_{n\in\mathbb{N}} \mathbb{E}_\pi[f(X_n)^+] = \pi(f^+) < \infty$, so that it converges $\mathbb{P}_\pi$-almost surely.

We argue by contradiction. If f is not $\mathbb{P}_\pi$-almost surely constant, then there exist $a < b$ such that $\pi(f < a) > 0$ and $\pi(f > b) > 0$. Then, $\mathbb{P}_\pi$-almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{f(X_k)<a\}} = \pi(f < a) > 0.$$

Hence $\#\{k : f(X_k) < a\} = \infty$, $\mathbb{P}_\pi - a.s.$, and similarly $\#\{k : f(X_k) > b\} = \infty$, $\mathbb{P}_\pi - a.s.$, which contradicts the almost sure convergence of $\{f(X_n)\ :\ n \in \mathbb{N}\}$. ■