▶ **M2DS: MCMC and Variational Inference**

1 Cours 1: Markov chains, transition kernel, invariant probability measure

(1) **Transition kernel, definitions** MULTIPLE CHOICE | Multiple answers allowed

Recall that $\mu P(A) = \int \mu(\mathrm{d}x) P(x, \mathrm{d}y) \mathbf{1}_A(y)$ and $Ph(x) = \int P(x, \mathrm{d}y) h(y)$. Let P a Markov kernel and let μ a probability measure on $(\mathsf{X}, \mathcal{X})$. Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with transition kernel P.

Which of the following answers are true?

a. $\mathbb{P}(X_{n+1} \in A | X_n) = P(X_{n+1}, A)$
b. If π is invariant for P and $X_0 \sim \pi$, then the conditional law of X_n given X_0 is π.
c. $\mathbb{E}[h(X_{n+2})|X_n] = P^2 h(X_n)$
d. If π is invariant for P and $X_0 \sim \pi$, then the conditional law of X_n given X_0 is π.
e. $\mu P^2(\mathsf{A}) = \int \cdots \int \mu(\mathrm{d}x_0) P(x_0, \mathrm{d}x_1) P(x_1, \mathrm{d}x_2) \mathbf{1}_{\mathsf{A}}(x_1)$
f. $\mu P^2(\mathsf{A}) = \int \cdots \int \mu(\mathrm{d}x_0) P(x_0, \mathrm{d}x_1) P(x_1, \mathrm{d}x_2) \mathbf{1}_{\mathsf{A}}(x_2)$
g. $\mathbb{E}[h(X_n)|X_{n-1}] = Ph(X_n)$
h. $A \mapsto P^2(x, A)$ is the law of X_2 given that $X_0 = x$.
i. μP^3 is the law of X_3 given that $X_0 \sim \mu$.

(2) **Expression of a transition kernel** MULTIPLE CHOICE | Multiple answers allowed

Let (X_k) be the Markov chain defined by the transition: given X_{k-1}, we draw independently $\epsilon_k \sim N(0,1)$ and a Bernoulli random variable Z_k with success probability α. Then,

- If $Z_k = 0$, we set $X_k = 2X_{k-1} + \epsilon_k$
- If $Z_k = 1$, we set $X_k = -X_{k-1} + 2\epsilon_k$

Denote by P the Markov kernel associated to (X_k). Let $\phi(x) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}$ be the density of a standard normal distribution.

Which of the following answers are true?

a.

$$\mathbb{E}[h(X_k)|X_{k-1}=x] = \alpha \int h(2x+\epsilon)\phi(\epsilon)\mathrm{d}\epsilon + (1-\alpha)\int h(-x+2\epsilon)\phi(\epsilon)\mathrm{d}\epsilon$$

b.

$$\mathbb{E}[h(X_k)|X_{k-1}=x] = (1-\alpha)\int h(2x+\epsilon)\phi(\epsilon)\mathrm{d}\epsilon + \alpha\int h(-x+2\epsilon)\phi(\epsilon)\mathrm{d}\epsilon$$

c.

$$\mathbb{E}[h(X_k)|X_{k-1}=x] = (1-\alpha)\int \phi(2x+\epsilon)h(\epsilon)\mathrm{d}\epsilon + \alpha\int \phi(-x+2\epsilon)h(\epsilon)\mathrm{d}\epsilon$$

d. $P(x,\mathrm{d}y) = \left[(1-\alpha)\phi(y-2x) + \frac{\alpha}{2}\phi((y+x)/2)\right]\mathrm{d}y$

e. $P(x,\mathrm{d}y) = \left[(1-\alpha)\phi(y-2x) + 2\alpha\phi(y+x)\right]\mathrm{d}y$

(3) **Expression of a transition kernel with degeneracy** MULTIPLE CHOICE

Multiple answers allowed

Let (X_k) be the Markov chain defined by the transition: given X_{k-1}, we draw independently $\epsilon_k \sim N(0,1)$ and a Bernoulli random variable Z_k with success probability α. Then,

- If $Z_k = 0$, we set $X_k = 2X_{k-1}$
- If $Z_k = 1$, we set $X_k = -X_{k-1} + 2\epsilon_k$

Denote by P the Markov kernel associated to (X_k). Let $\phi(x) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}$ be the density of a standard normal distribution.

Which of the following answers are true?

a.

$$\mathbb{E}[h(X_k)|X_{k-1}=x] = (1-\alpha)h(2x) + \alpha\int h(-x+2\epsilon)\phi(\epsilon)\mathrm{d}\epsilon$$

b. $P(x,\mathrm{d}y) = \left[(1-\alpha)\mathbf{1}\{y \neq 2x\} + 2\alpha\phi(y+x)\right]\mathrm{d}y$

c.

$$\mathbb{E}[h(X_k)|X_{k-1}=x] = \alpha h(2x) + (1-\alpha)\int h(-x+2\epsilon)\phi(\epsilon)\mathrm{d}\epsilon$$

d. $P(x,\mathrm{d}y) = (1-\alpha)\delta_{2x}(\mathrm{d}y) + \frac{\alpha}{2}\phi(\frac{y+x}{2})\mathrm{d}y$

Total of marks: 6