

► M2DS: MCMC and Variational Inference

# 1 Cours 1: Markov chains, transition kernel, invariant probability measure

## (1) Transition kernel, definitions

MULTIPLE CHOICE marked out of 2 penalty 1 Multiple answers allowed Shuffle

Recall that  $\mu P(A) = \int \mu(dx) P(x, dy) \mathbf{1}_A(y)$  and  $Ph(x) = \int P(x, dy) h(y)$ . Let  $P$  a Markov kernel and let  $\mu$  a probability measure on  $(X, \mathcal{X})$ . Let  $(X_n)_{n \in \mathbb{N}}$  be a Markov chain with transition kernel  $P$ .

Which of the following answers are true?

- a.  $A \mapsto P^2(x, A)$  is the law of  $X_2$  given that  $X_0 = x$ . (20%)
- b.  $\mu P^2(A) = \int \cdots \int \mu(dx_0) P(x_0, dx_1) P(x_1, dx_2) \mathbf{1}_A(x_1)$  (-20%)
- c.  $\mu P^2(A) = \int \cdots \int \mu(dx_0) P(x_0, dx_1) P(x_1, dx_2) \mathbf{1}_A(x_2)$  (20%)
- d.  $\mu P^3$  is the law of  $X_3$  given that  $X_0 \sim \mu$ . (20%)
- e.  $\mathbb{E}[h(X_{n+2})|X_n] = P^2 h(X_n)$  (20%)
- f.  $\mathbb{E}[h(X_n)|X_{n-1}] = Ph(X_{n-1})$  (-20%)
- g.  $\mathbb{P}(X_{n+1} \in A|X_n) = P(X_n, A)$  (-20%)
- h. If  $\pi$  is invariant for  $P$  and  $X_0 \sim \pi$ , then the conditional law of  $X_n$  given  $X_0$  is  $\pi$ . (-20%)
- i. If  $\pi$  is invariant for  $P$  and  $X_0 \sim \pi$ , then the conditional law of  $X_n$  given  $X_0$  is  $\pi$ . (20%)

## (2) Expression of a transition kernel

MULTIPLE CHOICE marked out of 2 penalty 1 Multiple answers allowed Shuffle

Let  $(X_k)$  be the Markov chain defined by the transition: given  $X_{k-1}$ , we draw independently  $\epsilon_k \sim N(0, 1)$  and a Bernoulli random variable  $Z_k$  with success probability  $\alpha$ . Then,

- If  $Z_k = 0$ , we set  $X_k = 2X_{k-1} + \epsilon_k$
- If  $Z_k = 1$ , we set  $X_k = -X_{k-1} + 2\epsilon_k$

Denote by  $P$  the Markov kernel associated to  $(X_k)$ . Let  $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$  be the density of a standard normal distribution.

Which of the following answers are true?

a.

$$\mathbb{E}[h(X_k)|X_{k-1} = x] = \alpha \int h(2x+\epsilon)\phi(\epsilon)d\epsilon + (1-\alpha) \int h(-x+2\epsilon)\phi(\epsilon)d\epsilon$$

(-50%)

b.

$$\mathbb{E}[h(X_k)|X_{k-1} = x] = (1-\alpha) \int h(2x+\epsilon)\phi(\epsilon)d\epsilon + \alpha \int h(-x+2\epsilon)\phi(\epsilon)d\epsilon$$

(50%)

c.

$$\mathbb{E}[h(X_k)|X_{k-1} = x] = (1-\alpha) \int \phi(2x+\epsilon)h(\epsilon)d\epsilon + \alpha \int \phi(-x+2\epsilon)h(\epsilon)d\epsilon$$

(-50%)

d.  $P(x, dy) = [(1-\alpha)\phi(y-2x) + \frac{\alpha}{2}\phi((y+x)/2)] dy$  (50%)

e.  $P(x, dy) = [(1-\alpha)\phi(y-2x) + 2\alpha\phi(y+x)] dy$  (-50%)

### (3) Expression of a transition kernel with degeneracy

MULTIPLE CHOICE

marked out of 2

penalty 1

Multiple answers allowed

Shuffle

Let  $(X_k)$  be the Markov chain defined by the transition: given  $X_{k-1}$ , we draw independently  $\epsilon_k \sim N(0, 1)$  and a Bernoulli random variable  $Z_k$  with success probability  $\alpha$ . Then,

- If  $Z_k = 0$ , we set  $X_k = 2X_{k-1}$
- If  $Z_k = 1$ , we set  $X_k = -X_{k-1} + 2\epsilon_k$

Denote by  $P$  the Markov kernel associated to  $(X_k)$ . Let  $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$  be the density of a standard normal distribution.

Which of the following answers are true?

a.

$$\mathbb{E}[h(X_k)|X_{k-1} = x] = (1-\alpha)h(2x) + \alpha \int h(-x+2\epsilon)\phi(\epsilon)d\epsilon$$

(50%)

b.

$$\mathbb{E}[h(X_k)|X_{k-1} = x] = \alpha h(2x) + (1 - \alpha) \int h(-x + 2\epsilon)\phi(\epsilon)d\epsilon$$

(-50%)

c.  $P(x, dy) = (1 - \alpha)\delta_{2x}(dy) + \frac{\alpha}{2}\phi(\frac{y+x}{2})dy$  (50%)

d.  $P(x, dy) = [(1 - \alpha)\mathbf{1}\{y \neq 2x\} + 2\alpha\phi(y + x)] dy$  (-50%)

*Total of marks: 6*