$\boxed{\text{Ex 4}}$ $X \perp\!\!\!\perp Y$. $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$.

Loi de $Z = \frac{X}{Y}$?

__Dém:__ Soit $h : \mathbb{R} \to \mathbb{R}$. C^0 bornée. On veut écrire : $\mathbb{E}(h(Z)) = \int h(z)\, f_Z(z)\, dz$.

$$\mathbb{E}(h(Z)) = \mathbb{E}\left(h\left(\frac{X}{Y}\right)\right) = \iint_{\mathbb{R}^2} h\left(\frac{x}{y}\right) f_{X,Y}(x,y)\, dx\, dy = \iint_{\mathbb{R}^2} h\left(\frac{x}{y}\right) \cdot \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dx\, dy.$$

$$A = \iint_{\mathbb{R}^2} h\left(\frac{x}{y}\right) \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}\, dx\, dy = \iint h(z) \frac{e^{-\frac{(1+z^2)u^2}{2}}}{2\pi} \underbrace{\left|\det\left(\frac{\partial(x,y)}{\partial(u,z)}\right)\right|}_{|u|} du\, dz.$$

Changement de variable : $(x,y) \longmapsto (z,u)$. où $\begin{cases} z = \frac{x}{y} \\ u = y \end{cases}$ i.e. $\begin{cases} x = uz \\ y = u \end{cases}$

$$\left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial z} \end{pmatrix}\right| = \left|\det\begin{pmatrix} z & u \\ 1 & 0 \end{pmatrix}\right| = |u|.$$

$dx\,dy = \left|\det \frac{\partial(u,z)}{\partial(x,y)}\right|^{-1}_{(u,z)} du\, dz$.

$$\left|\det\left(\frac{\partial(u,z)}{\partial(x,y)}\right)\right| = \left|\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 0 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix}\right|$$
$$= \left|\frac{1}{y}\right| = \left|\frac{1}{u}\right|.$$

$$A = \iint_{\mathbb{R}^2} h(z) \frac{e^{-(1+z^2)u^2/2}}{2\pi} |u|\, du\, dz$$
$$= \int h(z) \underbrace{\left(\int_{\mathbb{R}} \frac{e^{-\frac{(1+z^2)u^2}{2}}}{2\pi} |u|\, du\right)}_{f_Z(z)} dz. \quad (\text{par Fubini}).$$

$$\text{Alors : } f_Z(z) = \cancel{2} \int_0^{+\infty} \frac{e^{-\frac{(1+z^2)u^2}{2}}}{\cancel{2}\pi} u = \frac{1}{\pi}\left[\frac{e^{-(1+z^2)\frac{u^2}{2}}}{-(1+z^2)}\right]_{u=0}^{u\to\infty} = \frac{1}{\pi}\left(0 - \frac{1}{-(1+z^2)}\right)$$

$$f_Z(z) = \frac{1}{\pi(1+z^2)}.$$

donc : $\boxed{Z \text{ suit une loi de Cauchy.}}$

$\boxed{\text{Ex 5 :}}$

Rappel : La fonction Γ est définie par : $\Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x}\,dx$. $\left(\underset{\in \mathbb{R}}{a>0} \right)$

- $\forall a>1 \quad \Gamma(a) = (a-1)\,\Gamma(a-1)$. (Intégration par partie)

$\Gamma(1) = 1$.

donc : $\forall n \in \mathbb{N}^*, \ \Gamma(n) = (n-1)!$

On a : $1 = \int_0^{+\infty} \frac{x^{a-1} e^{-x}}{\Gamma(a)}\,dx = \int_{-\infty}^{+\infty} \underbrace{\frac{\theta^a x^{a-1} e^{-\theta x}}{\Gamma(a)} \mathbb{1}_{\mathbb{R}^+}(x)}_{\text{densité d'une loi } \Gamma(a,\theta)} dx$.

Rq : $\Gamma(1,\theta)$ = loi expon. de param. θ.

Soient X, Y v.a. ⫫, tq : $X \sim \Gamma(a,\theta)$, $Y \sim \Gamma(b,\theta)$.

Quelle est la loi de $\left(X+Y, \frac{X}{X+Y} \right) = (S,T)$?

Soit $h : \mathbb{R}^2 \to \mathbb{R}$ C^0 bornée (méthode de la fonction muette).

$$A = \mathbb{E}\left[h(S,T) \right] = \mathbb{E}\left[h\left(X+Y, \frac{X}{X+Y} \right) \right] = \iint_{\mathbb{R}^2} \mathbb{1}(x>0)\,\mathbb{1}(y>0)\, h\left(x+y, \frac{x}{x+y} \right) \frac{\theta^a x^{a-1} e^{-\theta x}}{\Gamma(a)} \times \frac{\theta^b y^{b-1} e^{-\theta y}}{\Gamma(b)}\,dx\,dy.$$

$$\underset{\text{Changement de variable}}{=} \iint \mathbb{1}(s>0, 0<t<1)\, h(s,t) \frac{\theta^{a+b}}{\Gamma(a)\Gamma(b)} (st)^{a-1} e^{-\theta s t} s^{b-1} (1-t)^{b-1} e^{-\theta s(1-t)} \underbrace{|s|}_{s} \,ds\,dt.$$

Changement de variable $(x,y) \mapsto (s,t)$ où $\begin{cases} s = x+y \\ t = \frac{x}{x+y} \end{cases}$ ie : $\begin{cases} x = st \\ y = s(1-t) \end{cases}$.

$$\mathbb{1}(x>0, y>0) = \mathbb{1}(\underbrace{st>0, s(1-t)>0}) = \mathbb{1}(\underbrace{s>0, st>0, s(1-t)>0}) + \mathbb{1}(s<0, st>0, \underset{<0}{s}(\underset{>0}{1-t})>0)$$
$$= \mathbb{1}(s>0, 0<t<1).$$

$$\left| \det \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \right| = \left| \det \begin{pmatrix} t & s \\ 1-t & -s \end{pmatrix} \right| = \left| -st - s(1-t) \right| = |s|.$$

$$A = \iint \mathbb{1}(s>0, 0<t<1)\, h(s,t) \frac{\theta^{a+b}}{\Gamma(a)\Gamma(b)} (st)^{a-1} e^{-\theta s t} s^{b-1} (1-t)^{b-1} e^{-\theta s(1-t)} \underbrace{|s|}_{s} \,ds\,dt.$$

$$= \iint h(s,t)\, \underbrace{\frac{\mathbb{1}(s>0)\,\theta^{a+b} s^{a+b-1} e^{-\theta s}}{\Gamma(a+b)}}_{\text{densité d'une loi } \Gamma(a+b,\theta)} \ \underbrace{\mathbb{1}(0<t<1) \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1} (1-t)^{b-1}}_{\text{densité d'une loi } \beta(a,b)} \,ds\,dt.$$

Finalement : si $X \sim \Gamma(a,\theta)$, $Y \sim \Gamma(b,\theta)$ et $X \perp Y$.

$$\boxed{\begin{array}{ll} \text{Alors : } S = X+Y \sim \Gamma(a+b,\theta). & \text{et } X+Y \perp\!\!\!\perp \dfrac{X}{X+Y}. \\ T := \dfrac{X}{X+Y} \sim \beta(a,b). & \end{array}}$$

Application : (a) si $X_1, \dots, X_n$ sont iid $X_1 \sim \exp(\lambda)$. alors : $X_1 + \dots + X_n \sim \Gamma(n,\lambda)$.

(b) Si $T \sim \beta(a,b)$. on sait que : $\int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1}(1-t)^{b-1}\,dt = 1$.

$$\mathbb{E}(T) = \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (t) \times t^{a-1}(1-t)^{b-1}\,dt. \quad (*)$$

(si on veut montrer $(*)$: $\underbrace{\Gamma(a+b)}_{\int_0^{+\infty} s^{a+b-1}e^{-s}\,ds} \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \underbrace{\Gamma(a)\,\Gamma(b)}_{\iint_{(\mathbb{R}^+)^2} u^{a-1}e^{-u}\,v^{b-1}e^{-v}\,du\,dv}$).

$x = s \quad y = st$.

$$\mathbb{E}(T) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \underbrace{\left(\int_0^1 t^{(a+1)-1}(1-t)^{b-1}\,dt \cdot \frac{\Gamma(a+1+b)}{\Gamma(a+1)\Gamma(b)}\right)}_{=1.\ (\text{par } *)} \times \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+1+b)}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cancel{\Gamma(b)}} \times \frac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+1+b)} = \frac{\Gamma(a+1)}{\Gamma(a)} \times \frac{\Gamma(a+b)}{\Gamma(a+b+1)} = \boxed{\frac{a}{a+b} = \mathbb{E}[T]}$$

Ex 2 : Soient (X_i) iid $X_i \sim \exp(\lambda)$. $\quad \lambda^n e^{-\lambda(x_1 + \dots + x_n)}$.

$(X_1 \dots X_n) \longrightarrow (X_1, \dots X_{n-1}, Z_n)$

Soient $Y_n = \max(X_1, \dots, X_n)$. et $Z_n = X_1 + \frac{X_2}{2} + \dots + \frac{X_n}{n}$. Mq : $Y_n \overset{\mathcal{L}}{=} Z_n$.

Dém : $\forall t \geq 0$. $\quad (X_i)$ iid.

$$\mathbb{P}(Y_n \leq t) = \mathbb{P}(X_1 \leq t, \dots, X_n \leq t) \overset{\Downarrow}{=} \prod_{i=1}^n \mathbb{P}(X_i \leq t) = \mathbb{P}(X_1 \leq t)^n.$$

$= (1 - e^{-\lambda t})^n$. et si $t < 0$, $\mathbb{P}(Y_n \leq t) = 0$.

Donc : Y_n a pour densité : $t \mapsto \underbrace{\mathbb{1}_{\mathbb{R}^+}(t)\lambda n e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}}_{f_{Y_n}(t)}$.

$$Z_n = X_1 + \frac{X_2}{2} + \dots + \frac{X_{n-1}}{n-1} + \frac{X_n}{n} = \underbrace{Z_{n-1}}_{} + \underbrace{\frac{X_n}{n}}_{=U_n}$$

$$\boxed{f_{Z_n} = f_{Z_{n-1}} * f_{U_n}}$$ (produit de convolution).

(Si $\left| \begin{array}{l} U \\ V \end{array} \right.$ a pour densité $\begin{array}{l} f_U \\ f_V \end{array}$. Posons $S = U + V$.

$U \perp\!\!\!\perp V$? densité de S ?

soit $h : \mathbb{R} \to \mathbb{R}$. C^0 bornée.

$$\mathbb{E}[h(S)] = \mathbb{E}(h(U+V)) = \iint h(u+v) f_U(u) f_V(v) \, du \, dv.$$

$$= \iint h(s) \, f_U(t) \, f_V(s-t) \left| \det \begin{pmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial s} \end{pmatrix} \right| ds \, dt.$$

Chgt variable. $(u,v) \mapsto (t, s)$ où : $\left| \begin{array}{l} t = u \\ s = u+v \end{array} \right.$ i.e $\left| \begin{array}{l} u = t \\ v = s - t \end{array} \right.$

$$\mathbb{E}(h(S)) = \iint h(s) \, f_U(t) f_V(s-t) \underbrace{\left| \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right|}_{1} ds \, dt.$$

$$= \int h(s) \cdot \Big[\underbrace{\int f_U(t) \, f_V(s-t) \, dt}_{f_U * f_V(s)} \Big] ds.$$

On suppose : $Z_{n-1} \overset{\mathcal{L}}{=} Y_{n-1}$. (Récurrence). $X_n \sim \exp(\lambda)$. $U_n = \frac{X_n}{n}$.

$$f_{Z_n}(x) = \int_{-\infty}^{+\infty} \underbrace{f_{Z_{n-1}}(y)}_{} \, f_{\frac{X_n}{n}}(x-y) \, dy.$$

$$= f_{Y_{n-1}}(y) = \lambda(n-1) e^{-\lambda y} (1 - e^{-\lambda y})^{n-2} \mathbb{1}_{y>0}.$$

$$\mathbb{E}(h(U_n)) = \int_0^{+\infty} h\left(\frac{x}{n}\right) \lambda e^{-\lambda x} dx = \int_0^{+\infty} h(t) \underbrace{\lambda e^{-\lambda n t} n \, dt}_{}$$

$$= \int_0^{+\infty} \lambda (n-1) \underbrace{e^{-\lambda y}}_{} (1 - e^{-\lambda y})^{n-2} \times \lambda e^{-\lambda n (x-y)} n \times \mathbb{1}_{(x-y)>0} \, dy.$$

$$= \lambda \quad e^{-\lambda n x} \times n . \underbrace{\int_0^x \underbrace{e^{-\lambda y}}_{} (1 - e^{-\lambda y})^{n-2} \underbrace{e^{\lambda y n}}_{} \, dy}_{\int_0^x (n-1) \lambda . \left(\underbrace{e^{\lambda y} - 1}_{}\right)^{n-2} \underbrace{e^{\lambda y} \, dy}_{\text{dfn } 1}}$$

$u = e^{\lambda y}$.

$$\underbrace{}_{\int_1^{e^{\lambda x}} (n-1)(u-1)^{n-2} du = \left[(u-1)^{n-1} \right]_1^{e^{\lambda x}} = (e^{\lambda x} - 1)^{n-1}.}$$

$$= \lambda \, \underbrace{e^{-\lambda n x}} \times n \times (e^{\lambda x} - 1)^{n-1}. \qquad (x>0).$$

$$= \lambda n \, e^{-\lambda x} \left(1 - e^{-\lambda x} \right)^{n-1}. \qquad (x>0).$$

$$= \mathbb{1}_{\mathbb{R}^+}(x) \lambda n \, e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{n-1}$$

D'où l'in fine : $\boxed{Z_n \overset{\mathcal{L}}{=} Y_n.}$ ce qui achève la preuve.