Problem

Let

- $\pi(\mathrm{d}y) = \pi(y)\mathrm{d}y$ be probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. As stressed by the expression $\pi(\mathrm{d}y) = \pi(y)\mathrm{d}y$, we assume that the measure π has a density on $\mathbb{R}$ with respect to the Lebesgue measure and by abuse of notation, we will also call π this density.

- $\phi(\mathrm{d}y) = \phi(y)\mathrm{d}y$ be another probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Again, the expression $\phi(\mathrm{d}y) = \phi(y)\mathrm{d}y$ means that we assume that the measure ϕ has a density on $\mathbb{R}$ with respect to the Lebesgue measure and by abuse of notation, we will also call ϕ this density.

In all the exercise, we assume that we can draw according to ϕ and that there exists a constant $\varepsilon > 0$ such that

$$\boxed{(A1) \qquad \forall x \in \mathbb{R}, \quad \pi(x) > \varepsilon\phi(x) > 0}$$

We now construct a family of random variables $(Z_t)_{t \geqslant 0}$ in the following way.

input : n
output: $Z_0, \ldots, Z_n$

At $t = 0$, draw $Z_0 \sim \mu$ where μ is arbitrary
for $t \leftarrow 1$ **to** n **do**

- Draw independently, $U_t \sim \mathrm{Unif}(0,1)$ and $Y_t \sim \phi$
- Letting $\beta : \mathbb{R} \to]0,1[$ be the function $\beta = \varepsilon\phi/\pi$, we set $Z_t = \begin{cases} Z_{t-1} & \text{if } U_t > \beta(Z_{t-1}) \\ Y_t & \text{if } U_t \leqslant \beta(Z_{t-1}) \end{cases}$

end

QUESTIONS

1. For any bounded measurable function $h : \mathbb{R} \to \mathbb{R}$, write $\mathbb{E}[h(Z_t)|Z_{t-1}]$. Deduce the expression of the Markov kernel P_1 associated to the Markov chain $(Z_k)_{k\in\mathbb{N}}$.

2. Show that the Markov kernel P_1 is π-reversible.

3. Show that π is the unique invariant probability measure for P_1.

4. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function such that $P_1 h = h$. Then, show that h is constant.

5. Let $(Z_k)_{k\in\mathbb{N}}$ be a Markov chain with Markov kernel P_1. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\pi(|f|) < \infty$. Define
$$A = \left\{ \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} f(Z_k) = \pi(f) \right\}$$
Setting $h(x) = \mathbb{E}_x[\mathbf{1}_{A^c}] = \mathbb{P}_x(A^c)$, we admit that $P_1 h = h$ (it is actually proved in the Lecture Notes). Deduce from the previous question that the Law of Large Numbers holds for $(Z_k)_{k\in\mathbb{N}}$ starting from any initial distribution, i.e. for any probability measure ξ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,
$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} f(Z_k) = \pi(f), \quad \mathbb{P}_\xi - a.s.$$

We now let

- $Q(x,\mathrm{d}y) = q(x,y)\mathrm{d}y$ be a Markov kernel on $\mathbb{R}\times\mathcal{B}(\mathbb{R})$. Thus, we assume that Q admits the Markov kernel density q with respect to the Lebesgue measure.
- P_0 be the Markov kernel associated to a "classical" Metropolis-Hastings algorithm, with proposal kernel Q and target distribution π, that is for any $x\in\mathbb{R}$,

$$P_0(x,\mathrm{d}y) = Q(x,\mathrm{d}y)\alpha(x,y) + \bar{\alpha}(x)\delta_x(\mathrm{d}y)$$

  where for any $x,y\in\mathbb{R}$,

$$\alpha(x,y) = \min\left(\frac{\pi(y)q(y,x)}{\pi(x)q(x,y)},1\right), \quad \bar{\alpha}(x) = 1-\int Q(x,\mathrm{d}z)\alpha(x,z)$$

In addition to Assumption $(A1)$, we now also assume

$$\boxed{(A2) \qquad \forall x,y\in\mathbb{R}, \quad q(x,y)>0}$$

We now construct a family of random variables $(X_t)_{t\geqslant 0}$ in the following way.

input : n
output: $X_0,\dots,X_n$

At $t=0$, draw $X_0\sim\mu$ where μ is arbitrary
for $t\leftarrow 1$ **to** n **do**

- Draw independently, $X_t'\sim P_0(X_{t-1},\cdot)$, $U_t\sim\mathrm{Unif}(0,1)$ and $Y_t\sim\phi$
- Letting $\beta:\mathbb{R}\to]0,1[$ be the function $\beta=\varepsilon\phi/\pi$, we set $X_t = \begin{cases} X_t' & \text{if } U_k > \beta(X_k') \\ Y_t & \text{if } U_k \leqslant \beta(X_k') \end{cases}$

end

QUESTIONS (CONTINUED)

6. For any bounded measurable function h, write $\mathbb{E}[h(X_t)|X_{t-1}]$ in terms of P_0,β and ϕ. Deduce that there exists functions γ_0 and γ_1 such that the Markov kernel P_2 associated to the Markov chain $(X_k)_{k\in\mathbb{N}}$ can be written as
$$P_2(x,\mathrm{d}y) = P_0(x,\mathrm{d}y)\gamma_0(y) + \gamma_1(x)\phi(y)\mathrm{d}y$$
and give the expressions of the functions γ_0 and γ_1.
7. Check that $P_2 = P_0P_1$
8. Show that π is invariant for the Markov kernel P_2.
9. Can we say that π is the unique invariant probability distribution for P_2?
10. **(More Difficult)** Show that the Law of Large Numbers for $(X_t)_{t\geqslant 0}$ holds starting from any initial distribution ξ.