Chapter 4
Exercices Week 4

4.1 Dirichlet and Poisson problems

Definition 4.1 (Dirichlet Problem) *Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$, $A \in \mathscr{X}$ and $g \in \mathbb{F}_+(\mathsf{X})$. A nonnegative function $u \in \mathbb{F}_+(\mathsf{X})$ is a solution to the Dirichlet problem if*

$$u(x) = \begin{cases} g(x) , & x \in A , \\ Pu(x) , & x \in A^c . \end{cases} \tag{4.1}$$

For $A \in \mathscr{X}$, we define a submarkovian kernel P_A for $x \in \mathsf{X}$ and $B \in \mathscr{X}$ by

$$P_A(x,B) = \mathbb{E}_x[\mathbb{1}_{\{\tau_A<\infty\}}\mathbb{1}_B(X_{\tau_A})] = \mathbb{P}_x(\tau_A < \infty, X_{\tau_A} \in B) , \tag{4.2}$$

which is the probability that the chain starting from x eventually hits the set $A \cap B$.

4.1. For any $A \in \mathscr{X}$ and $g \in \mathbb{F}_+(\mathsf{X})$, the function $P_A g$ is a solution to the Dirichlet problem (4.1)

Definition 4.2 (Poisson problem) *Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$, $A \in \mathscr{X}$ and $f : A^c \to \mathbb{R}_+$ be a measurable function. A nonnegative function $u \in \mathbb{F}_+(\mathsf{X})$ is a solution to the Poisson problem if*

$$u(x) = \begin{cases} 0 , & x \in A , \\ Pu(x) + f(x) , & x \in A^c . \end{cases} \tag{4.3}$$

For $A \in \mathscr{X}$ and $h \in \mathbb{F}_+(\mathsf{X})$ define

$$G_A h(x) = \mathbb{1}_{A^c}(x)\mathbb{E}_x\left[\sum_{k=0}^{\tau_A-1} h(X_k)\right] = \mathbb{E}_x\left[\sum_{k=0}^{\tau_A-1} h(X_k)\right] , \tag{4.4}$$

where we have used the convention $\sum_{k=0}^{-1} \cdot = 0$. Note that $G_A h$ is nonnegative but we do not assume that it is finite.

4.2. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$, $A \in \mathscr{X}$ and $f : A^c \to \mathbb{R}_+$ be a measurable function. The function $G_A f$ is a solution to the Poisson problem (4.3).

Solutions to exercises

4.1 If $x \in A$, then by definition, $P_A g(x) = g(x)$. For $x \in \mathsf{X}$, the identity $\sigma_A = 1 + \tau_A \circ \theta_1$ and the Markov property yield

$$\begin{aligned} PP_A g(x) &= \mathbb{E}_x[P_A g(X_1)] = \mathbb{E}_x[\{\mathbb{1}_{\{\tau_A < \infty\}} g(X_{\tau_A})\} \circ \theta_1] \\ &= \mathbb{E}_x[\mathbb{1}_{\{\tau_A \circ \theta_1 < \infty\}} g(X_{1+\tau_A \circ \theta_1})] = \mathbb{E}_x[\mathbb{1}_{\{\sigma_A < \infty\}} g(X_{\sigma_A})] \;. \end{aligned}$$

For $x \notin A$, then $\sigma_A = \tau_A$ $\mathbb{P}_x -$a.s. and we obtain

$$PP_A g(x) = \mathbb{E}_x[\mathbb{1}_{\{\tau_A < \infty\}} g(X_{\tau_A})] = P_A g(x) \;.$$

4.2 Set $u(x) = G_A f(x) = \mathbb{E}_x[S]$ where $S = \mathbb{1}_{A^c}(X_0) \sum_{k=0}^{\tau_A - 1} f(X_k)$. By convention $u(x) = 0$ for $x \in A$. Applying the Markov property and the relation $\sigma_A = 1 + \tau_A \circ \theta_1$, we obtain

$$\begin{aligned} Pu(x) &= \mathbb{E}_x[u(X_1)] = \mathbb{E}_x[\mathbb{E}_{X_1}[S]] = \mathbb{E}_x[\mathbb{E}_x[S \circ \theta_1 \mid \mathscr{F}_1]] \\ &= \mathbb{E}_x[S \circ \theta_1] = \mathbb{E}_x\left[\mathbb{1}_{A^c}(X_1) \sum_{k=1}^{\tau_A \circ \theta_1} f(X_k)\right] = \mathbb{E}_x\left[\sum_{k=1}^{\sigma_A - 1} f(X_k)\right] \;, \end{aligned} \tag{4.5}$$

where the last equality follows from $\mathbb{1}_A(X_1) \sum_{k=1}^{\sigma_A - 1} f(X_k) = 0$. For $x \notin A$, $\sigma_A = \tau_A$ $\mathbb{P}_x -$a.s. and thus

$$f(x) + Pu(x) = f(x) + \mathbb{E}_x\left[\sum_{k=1}^{\sigma_A - 1} f(X_k)\right] = \mathbb{E}_x\left[\mathbb{1}_{A^c}(X_0) \sum_{k=0}^{\tau_A - 1} f(X_k)\right] = u(x) \;.$$