Chapter 3
Exercices Week 3

Definition 3.1 *We define the hitting time and the return to a set* A *as:*

$$\tau_{\mathsf{A}} = \inf\{k \geqslant 0 : X_k \in \mathsf{A}\}$$
$$\sigma_{\mathsf{A}} = \inf\{k \geqslant 1 : X_k \in \mathsf{A}\} \ .$$

Similarly we define the n-th successive return times as

$$\sigma_{\mathsf{A}}^{(n)} = \inf\left\{k > \sigma_{\mathsf{A}}^{(n-1)} : X_k \in \mathsf{A}\right\} \quad , \quad \sigma_{\mathsf{A}}^{0} = 0 \ .$$

3.1. Let τ and σ be two stopping times with respect to the canonical filtration $(\mathscr{F}_n)_{n\geq 0}$. Show that

1. for any $n,m \in \mathbb{N}$, $\theta_m^{-1}(\mathscr{F}_n) = \sigma(X_m, \ldots, X_{n+m})$
2. Define the random variable
$$\sigma = \begin{cases} \tau_1 + \tau_0 \circ \theta_{\tau_1} \text{ if } \tau_1 < \infty \\ \infty \text{ otherwise .} \end{cases}$$
Show that σ is a stopping time and on $\{\tau_1 < \infty\} \cap \{\tau_0 < \infty\}$,
$$X_{\tau_0} \circ \theta_{\tau_1} = X_\sigma \ . \tag{3.1}$$

Let $\mathsf{A} \subset \mathscr{X}$.

3. Show that $\{\sigma_{\mathsf{A}}^{(n)}\}_n$ is a sequence of stopping times verifying for any n:
$$\sigma_{\mathsf{A}} = 1 + \tau_{\mathsf{A}} \circ \theta_1 \ , \tag{3.2}$$
$$\sigma_{\mathsf{A}}^{(n)} = \sigma_{\mathsf{A}}^{(n-1)} + \sigma_{\mathsf{A}} \circ \theta_{\sigma_{\mathsf{A}}^{(n-1)}} \qquad n \geq 1 \ . \tag{3.3}$$

3.2. Let $\mathsf{C} \subset \mathscr{X}$. Show that

1. If for any $x \in \mathsf{C}$, $\mathbb{P}_x(\sigma_{\mathsf{C}} < \infty) = 1$, then for any $n \geq 1$ and for any $x \in \mathsf{C}$, $\mathbb{P}_x(\sigma_{\mathsf{C}}^{(n)} < \infty) = 1$.
2. If for any $x \in \mathsf{C}^c$, $\mathbb{P}_x(\sigma_{\mathsf{C}} < \infty) = 1$, then for any $n \geq 1$ and for any $x \in \mathsf{X}$, $\mathbb{P}_x(\sigma_{\mathsf{C}}^{(n)} < \infty) = 1$.

Definition 3.2 *For any set* $C \in \mathscr{X}$*, denote by* $\mathscr{X}_C$ *the subset of* $\mathscr{X}$ *defined as*

$$\mathscr{X}_C = \{A \cap C : A \in \mathscr{X}\} \ . \tag{3.4}$$

It is easily seen that $\mathscr{X}_C$ is a σ-field, often called the trace σ-field on C or the induced σ-field on C.

Definition 3.3 (Induced kernel) *For all $C \in \mathscr{X}$, the induced kernel Q_C on $C \times \mathscr{X}_C$ is defined by*

$$Q_C(x,B) = \mathbb{P}_x(X_{\sigma_C} \in B,\ \sigma_C < \infty)\ , \qquad x \in C\,,\ B \in \mathscr{X}_C\ . \tag{3.5}$$

3.3. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$ and $C \in \mathscr{X}$. Assume that $\mathbb{P}_x(\sigma_C < \infty) = 1$ for all $x \in C$. Then, for all $x \in C$ and $n \in \mathbb{N}$, $\mathbb{P}_x(\sigma_C^{(n)} < \infty) = 1$. We set for all $n \in \mathbb{N}$,

$$\tilde{X}_n = X_{\sigma_C^{(n)}} \mathbb{1}_{\{\sigma_C^{(n)} < \infty\}} + x_* \mathbb{1}_{\{\sigma_C^{(n)} = \infty\}} \tag{3.6}$$

where x_* is an arbitrary element of C.

(i) Show that, for all $x \in C$, the process $\{\tilde{X}_n,\ n \in \mathbb{N}\}$ is under $\mathbb{P}_x$ a Markov chain on C with kernel Q_C (see Definition 3.3).
(ii) Let $A \subset C$ and denote by $\tilde{\sigma}_A$ the return time to the set A of the chain $\{\tilde{X}_n\}$. Show that, for all $x \in C$, $\mathbb{E}_x[\sigma_A] \le \mathbb{E}_x[\tilde{\sigma}_A] \sup_{y \in C} \mathbb{E}_y[\sigma_C]$.

3.4 (Maximum principle). Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Show that for all $x \in \mathsf{X}$ and $A \in \mathscr{X}$,

$$U(x,A) \le \mathbb{P}_x(\tau_A < \infty) \sup_{y \in A} U(y,A)\ .$$

3.5. Show that for every $A \in \mathscr{X}$, the function $x \mapsto \mathbb{P}_x(N_A = \infty)$ is harmonic.

3.6. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Let $A \in \mathscr{X}$.

(i) Assume that there exists $\delta \in [0,1)$ such that $\sup_{x \in A} \mathbb{P}_x(\sigma_A < \infty) \le \delta$. Show that for all $p \in \mathbb{N}^*$, $\sup_{x \in A} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \le \delta^p$ and $\sup_{x \in \mathsf{X}} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \le \delta^{p-1}$. Moreover,

$$\sup_{x \in \mathsf{X}} U(x,A) \le (1-\delta)^{-1}\ . \tag{3.7}$$

(ii) Assume that $\mathbb{P}_x(\sigma_A < \infty) = 1$ for all $x \in A$. Show that for all $p \in \mathbb{N}^*$, $\inf_{x \in A} \mathbb{P}_x(\sigma_A^{(p)} < \infty) = 1$. Moreover, $\inf_{x \in A} \mathbb{P}_x(N_A = \infty) = 1$ for all $x \in A$.

Given $A \in \mathscr{X}$, we define, for $n \ge 1$ and $B \in \mathscr{X}$,

$$_A^nP(x,B) = \mathbb{P}_x(X_n \in B\,,\ n \le \sigma_A)\ . \tag{3.8}$$

Thus $_A^nP(x,B)$ is the probability that the chain goes from x to B in n steps without visiting the set A. It is called the n-step taboo probability. Note that $_A^1P = P$ and $_A^nP = (PI_{A^c})^{n-1}P$ where I_A is the kernel defined by $I_A f(x) = \mathbb{1}_A(x) f(x)$ for any $f \in \mathbb{F}_+(\mathsf{X})$

3.7. 1. Show the *first-entrance decomposition*

$$P^n f(x) = {}_A^nPf(x) + \sum_{j=1}^{n-1} {}_A^jP(\mathbb{1}_A \times P^{n-j} f)(x)\ . \tag{3.9}$$

2. Show the *last exit decomposition*

$$P^n f(x) = {}_A^nPf(x) + \sum_{j=1}^{n-1} P^j(\mathbb{1}_A \times {}_A^{n-j}Pf)(x)\ . \tag{3.10}$$

3.8. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Let $A \in \mathscr{X}$.

1. Show that the following conditions are equivalent.
   (i) A is accessible.
   (ii) For every $x \in \mathsf{X}$, there exists an integer $n \geq 1$ such that $P^n(x,A) > 0$.
   (iii) For every $\mu \in \mathbb{M}_+(\mathscr{X})$, there exists an integer $n \geq 1$ such that $\mu P^n(A) > 0$.
   (iv) For every $x \in A^c$, $\mathbb{P}_x(\sigma_A < \infty) > 0$.
2. Show that, if A is accessible, for all $a \in \mathbb{M}_+^1(\mathbb{N})$ with $a(k) > 0$ for $k \geq 1$, $K_a(x,A) > 0$ for all $x \in \mathsf{X}$.
3. Show that if there exists $a \in \mathbb{M}_+^1(\mathbb{N})$ such that $K_a(x,A) > 0$ for all $x \in \mathsf{X}$, then A is accessible.

Definition 3.4 (Domain of attraction of a set, attractive set) *Let P be a Markov chain on $\mathsf{X} \times \mathscr{X}$. The domain of attraction C_+ of a non empty set $C \in \mathscr{X}$ is the set of states $x \in \mathsf{X}$ from which the Markov chain returns to C with probability one:*

$$C_+ = \{x \in \mathsf{X} : \mathbb{P}_x(\sigma_C < \infty) = 1\} . \tag{3.11}$$

(i) If $C \subset C_+$, then the set C is said to be Harris recurrent.
(ii) If $C_+ = \mathsf{X}$, then the set C is said to be attractive.

If the domain of attraction C_+ of C contains C, then it may happen that $C_+ \subsetneq \mathsf{X}$. Nevertheless, as shown below, the set C_+ is absorbing.

3.9. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Let $C \in \mathscr{X}$ be a non-empty set such that $C \subset C_+$. Show that the set C_+ is absorbing.

Solutions to exercises

3.3 (i) Let $x \in C$. Since $\mathbb{P}_x(\sigma_C^{(n)} < \infty) = 1$ for all $x \in C$ and $n \in \mathbb{N}$, the strong Markov property applied to the Markov chain $\{X_n\}$ yields, for any $B \in \mathscr{X}$,

$$\begin{aligned}\mathbb{P}_x\left(\tilde{X}_{n+1} \in B \,\middle|\, \mathscr{F}_{\sigma_C^{(n)}}\right) &= \mathbb{P}_x\left(X_{\sigma_C^{(n+1)}} \in B \,\middle|\, \mathscr{F}_{\sigma_C^{(n)}}\right) = \mathbb{P}_x\left(X_{\sigma_C} \circ \theta_{\sigma_C^{(n)}} \in B \,\middle|\, \mathscr{F}_{\sigma_C^{(n)}}\right) \\ &= \mathbb{P}_{X_{\sigma_C^{(n)}}}(X_{\sigma_C} \in B) = Q_C(\tilde{X}_n, B) \,.\end{aligned}$$

(ii) Since $A \subset C$, we have $\sigma_A = \sigma_C^{(\tilde{\sigma}_A)}$. Thus,

$$\sigma_A = \sum_{n=0}^{\tilde{\sigma}_A - 1} \{\sigma_C^{(n+1)} - \sigma_C^{(n)}\} = \sum_{n=0}^{\infty} \{\sigma_C^{(n+1)} - \sigma_C^{(n)}\} \mathbb{1}_{\{n < \tilde{\sigma}_A\}} = \sum_{n=0}^{\infty} \sigma_C \circ \theta_{\sigma_C^{(n)}} \mathbb{1}_{\{n < \tilde{\sigma}_A\}} \,.$$

Let $x \in C$. Note that $\{n < \tilde{\sigma}_A\} = \cap_{i=1}^{n} \{X_{\sigma^{(i)}} \notin A\} \in \mathscr{F}_{\sigma^{(n)}}$ and applying again **??**, we have $\mathbb{P}_x(\sigma_C^{(n)} < \infty) = 1$. We then obtain by the strong Markov property,

$$\begin{aligned}\mathbb{E}_x[\sigma_A] &= \sum_{n=0}^{\infty} \mathbb{E}_x[\sigma_C \circ \theta_{\sigma_C^{(n)}} \mathbb{1}\{n < \tilde{\sigma}_A\}] \\ &= \sum_{n=0}^{\infty} \mathbb{E}_x[\mathbb{1}\{n < \tilde{\sigma}_A\} \mathbb{E}_{X_{\sigma_C^{(n)}}}[\sigma_C]] \le \mathbb{E}_x[\tilde{\sigma}_A] \sup_{y \in C} \mathbb{E}_y[\sigma_C] \,.\end{aligned}$$

3.4 By the strong Markov property, we get

$$\begin{aligned}U(x,A) &= \mathbb{E}_x\left[\sum_{n=0}^{\infty} \mathbb{1}_A(X_n)\right] = \mathbb{E}_x\left[\sum_{n=\tau_A}^{\infty} \mathbb{1}_A(X_n) \mathbb{1}\{\tau_A < \infty\}\right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbb{1}_A(X_n \circ \theta_{\tau_A}) \mathbb{1}\{\tau_A < \infty\}\right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbb{1}\{\tau_A < \infty\} \mathbb{E}_{X_{\tau_A}}\left[\mathbb{1}_A(X_n)\right]\right] \le \mathbb{P}_x(\tau_A < \infty) \sup_{y \in A} U(y,A) \,.\end{aligned}$$

3.5 Define $h(x) = \mathbb{P}_x(N_A = \infty)$. Then $Ph(x) = \mathbb{E}_x[h(X_1)] = \mathbb{E}_x[\mathbb{P}_{X_1}(N_A = \infty)]$ and applying the Markov property, we obtain

$$Ph(x) = \mathbb{E}_x[\mathbb{P}_x(N_A \circ \theta = \infty \,|\, \mathscr{F}_1)] = \mathbb{P}_x(N_A \circ \theta = \infty) = \mathbb{P}_x(N_A = \infty) = h(x) \,.$$

3.6 (i) For $p \in \mathbb{N}$, $\sigma_A^{(p+1)} = \sigma_A^{(p)} + \sigma_A \circ \theta_{\sigma_A^{(p)}}$ on $\{\sigma_A^{(p)} < \infty\}$. Applying the strong Markov property yields

$$\begin{aligned}\mathbb{P}_x(\sigma_A^{(p+1)} < \infty) &= \mathbb{P}_x\left(\sigma_A^{(p)} < \infty, \sigma_A \circ \theta_{\sigma_A^{(p)}} < \infty\right) \\ &= \mathbb{E}_x\left[\mathbb{1}\left\{\sigma_A^{(p)} < \infty\right\} \mathbb{P}_{X_{\sigma_A^{(p)}}}(\sigma_A < \infty)\right] \le \delta \mathbb{P}_x(\sigma_A^{(p)} < \infty) \,.\end{aligned}$$

By induction, we obtain $\mathbb{P}_x(\sigma_A^{(p)} < \infty) \le \delta^p$ for every $p \in \mathbb{N}^*$ and $x \in A$. Thus, for $x \in A$,

$$U(x,A) = \mathbb{E}_x[N_A] \leq 1 + \sum_{p=1}^{\infty} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \leq (1-\delta)^{-1} .$$

Since by **??** for all $x \in \mathsf{X}$, $U(x,A) \leq \sup_{y \in A} U(y,A)$, (3.7) follows.

(ii) By **??**, $\mathbb{P}_x(\sigma_A^{(n)} < \infty) = 1$ for every $n \in \mathbb{N}$ and $x \in A$. Then,

$$\mathbb{P}_x(N_A = \infty) = \mathbb{P}_x\left(\bigcap_{n=1}^{\infty} \{\sigma_A^{(n)} < \infty\}\right) = 1 .$$

3.7 1. Using the Markov property,

$$\begin{aligned}
P^n f(x) = \mathbb{E}_x[f(X_n)] &= \mathbb{E}_x[\mathbb{1}\{n \leq \sigma_A\} f(X_n)] + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{\sigma_A = j\} f(X_n)] \\
&= {}_A^n P f(x) + \sum_{j=1}^{n-1} \mathbb{E}_x\left[\mathbb{1}\{\sigma_A = j\} \mathbb{E}_{X_j}[f(X_{n-j})]\right] \\
&= {}_A^n P f(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{\sigma_A \geq j\} \mathbb{1}_A(X_j) P^{n-j} f(X_j)] \\
&= {}_A^n P f(x) + \sum_{j=1}^{n-1} {}_A^j P(\mathbb{1}_A \times P^{n-j} f)(x) . \qquad (3.12)
\end{aligned}$$

2. The last exit decomposition is established analogously.

$$\begin{aligned}
P^n f(x) &= \mathbb{E}_x[f(X_n)] \\
&= \mathbb{E}_x[\mathbb{1}_{\{n \leq \sigma_A\}} f(X_n)] + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{X_j \in A, X_{j+1} \notin A, \ldots, X_{n-1} \notin A\} f(X_n)] \\
&= {}_A^n P f(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}_A(X_j) \mathbb{E}_{X_j}[\mathbb{1}\{X_1 \notin A, \ldots, X_{n-j-1} \notin A\} f(X_{n-j})]] \\
&= {}_A^n P f(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}_A(X_j)\, {}_A^{n-j} P f(X_j)] \\
&= {}_A^n P f(x) + \sum_{j=1}^{n-1} P^j(\mathbb{1}_A \times {}_A^{n-j} P f)(x) . \qquad (3.13)
\end{aligned}$$

3.8 The assertion (iv) $\Rightarrow$ (i) is the only non trivial one. It means that if A can be reached from A^c, then it can be reached from A. Indeed, starting from A, either the chain remains in A, or it leaves A and then can reach it again. Formally, applying the Markov property yields

$$\begin{aligned}
\mathbb{P}_x(\sigma_A < \infty) &= \mathbb{P}_x(X_1 \in A) + \mathbb{P}_x(X_1 \in A^c, \sigma_A \circ \theta < \infty) \\
&= \mathbb{P}_x(X_1 \in A) + \mathbb{E}_x[\mathbb{1}_{A^c}(X_1) \mathbb{P}_{X_1}(\sigma_A < \infty)] .
\end{aligned}$$

For each $x \in \mathsf{X}$, either $\mathbb{P}_x(X_1 \in A) > 0$ or $\mathbb{P}_x(X_1 \in A) = 0$. In the latter case, it then holds that $\mathbb{P}_x(\sigma_A < \infty) = \mathbb{E}_x[\mathbb{1}_{A^c}(X_1) \mathbb{P}_{X_1}(\sigma_A < \infty)] > 0$ if (iv) holds. Thus (iv) $\Rightarrow$ (i).

3.9 Let $x \in C_+$. Then,

$$
\begin{aligned}
0 = \mathbb{P}_x(\sigma_C = \infty) &\geq \mathbb{P}_x(X_1 \in C^c\,,\ \sigma_C \circ \theta = \infty) \\
&\geq \mathbb{P}_x(X_1 \in C^c_+\,,\ \sigma_C \circ \theta = \infty) = \mathbb{E}_x[\mathbb{1}_{C^c_+}(X_1)\mathbb{P}_{X_1}(\sigma_C = \infty)]\,.
\end{aligned}
$$

Since $\mathbb{P}_y(\sigma_C = \infty) > 0$ for $y \in C^c_+$, this yields $P(x, C^c_+) = \mathbb{P}_x(X_1 \in C^c_+) = 0$.