Chapter 2
Exercices Sheet 2

For any set $C \in \mathscr{X}$, denote by $\mathscr{X}_C$ the subset of $\mathscr{X}$ defined as

$$\mathscr{X}_C = \{A \cap C : A \in \mathscr{X}\} . \tag{2.1}$$

It is easily seen that $\mathscr{X}_C$ is a σ-field, often called the trace σ-field on C or the induced σ-field on C.

Definition 2.1 (Induced kernel) *For all $C \in \mathscr{X}$, the induced kernel Q_C on $C \times \mathscr{X}_C$ is defined by*

$$Q_C(x,B) = \mathbb{P}_x(X_{\sigma_C} \in B,\ \sigma_C < \infty) , \qquad x \in C ,\ B \in \mathscr{X}_C . \tag{2.2}$$

2.1. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$ and $C \in \mathscr{X}$. Assume that $\mathbb{P}_x(\sigma_C < \infty) = 1$ for all $x \in C$. Then, for all $x \in C$ and $n \in \mathbb{N}$, $\mathbb{P}_x(\sigma_C^{(n)} < \infty) = 1$. We set for all $n \in \mathbb{N}$,

$$\tilde{X}_n = X_{\sigma_C^{(n)}} \mathbb{1}_{\{\sigma_C^{(n)} < \infty\}} + x_* \mathbb{1}_{\{\sigma_C^{(n)} = \infty\}} \tag{2.3}$$

where x_* is an arbitrary element of C.

(i) Show that, for all $x \in C$, the process $\{\tilde{X}_n,\ n \in \mathbb{N}\}$ is under $\mathbb{P}_x$ a Markov chain on C with kernel Q_C (see Definition 2.1).
(ii) Let $A \subset C$ and denote by $\tilde{\sigma}_A$ the return time to the set A of the chain $\{\tilde{X}_n\}$. Show that, for all $x \in C$, $\mathbb{E}_x[\sigma_A] \le \mathbb{E}_x[\tilde{\sigma}_A] \sup_{y \in C} \mathbb{E}_y[\sigma_C]$.

2.2 (Maximum principle). Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Show that for all $x \in \mathsf{X}$ and $A \in \mathscr{X}$,

$$U(x,A) \le \mathbb{P}_x(\tau_A < \infty) \sup_{y \in A} U(y,A) .$$

2.3. Show that for every $A \in \mathscr{X}$, the function $x \mapsto \mathbb{P}_x(N_A = \infty)$ is harmonic.

2.4. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Let $A \in \mathscr{X}$.

(i) Assume that there exists $\delta \in [0,1)$ such that $\sup_{x \in A} \mathbb{P}_x(\sigma_A < \infty) \le \delta$. Show that for all $p \in \mathbb{N}^*$, $\sup_{x \in A} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \le \delta^p$ and $\sup_{x \in \mathsf{X}} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \le \delta^{p-1}$. Moreover,

$$\sup_{x \in \mathsf{X}} U(x,A) \le (1-\delta)^{-1} . \tag{2.4}$$

(ii) Assume that $\mathbb{P}_x(\sigma_A < \infty) = 1$ for all $x \in A$. Show that for all $p \in \mathbb{N}^*$, $\inf_{x \in A} \mathbb{P}_x(\sigma_A^{(p)} < \infty) = 1$. Moreover, $\inf_{x \in A} \mathbb{P}_x(N_A = \infty) = 1$ for all $x \in A$.

Given $A \in \mathscr{X}$, we define, for $n \geq 1$ and $B \in \mathscr{X}$,

$$ {}_A^n P(x,B) = \mathbb{P}_x(X_n \in B\,,\; n \leq \sigma_A)\;. \tag{2.5} $$

Thus ${}_A^n P(x,B)$ is the probability that the chain goes from x to B in n steps without visiting the set A. It is called the n-step taboo probability. Note that ${}_A^1 P = P$ and ${}_A^n P = (PI_{A^c})^{n-1}P$ where I_A is the kernel defined by $I_A f(x) = \mathbb{1}_A(x) f(x)$ for any $f \in \mathbb{F}_+(\mathsf{X})$

2.5. 1. Show the *first-entrance decomposition*

$$ P^n f(x) = {}_A^n P f(x) + \sum_{j=1}^{n-1} {}_A^j P(\mathbb{1}_A \times P^{n-j} f)(x)\;. \tag{2.6} $$

2. Show the *last exit decomposition*

$$ P^n f(x) = {}_A^n P f(x) + \sum_{j=1}^{n-1} P^j(\mathbb{1}_A \times {}^{n-j}_{A} P f)(x)\;. \tag{2.7} $$

2.6. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Let $A \in \mathscr{X}$.

1. Show that the following conditions are equivalent.
   (i) A is accessible.
   (ii) For every $x \in \mathsf{X}$, there exists an integer $n \geq 1$ such that $P^n(x,A) > 0$.
   (iii) For every $\mu \in \mathbb{M}_+(\mathscr{X})$, there exists an integer $n \geq 1$ such that $\mu P^n(A) > 0$.
   (iv) For every $x \in A^c$, $\mathbb{P}_x(\sigma_A < \infty) > 0$.
2. Show that, if A is accessible, for all $a \in \mathbb{M}_+^1(\mathbb{N})$ with $a(k) > 0$ for $k \geq 1$, $K_a(x,A) > 0$ for all $x \in \mathsf{X}$.
3. Show that if there exists $a \in \mathbb{M}_+^1(\mathbb{N})$ such that $K_a(x,A) > 0$ for all $x \in \mathsf{X}$, then A is accessible.

Definition 2.2 (Domain of attraction of a set, attractive set) *Let P be a Markov chain on $\mathsf{X} \times \mathscr{X}$. The domain of attraction C_+ of a non empty set $C \in \mathscr{X}$ is the set of states $x \in \mathsf{X}$ from which the Markov chain returns to C with probability one:*

$$ C_+ = \{x \in \mathsf{X} :\; \mathbb{P}_x(\sigma_C < \infty) = 1\}\;. \tag{2.8} $$

(i) If $C \subset C_+$, then the set C is said to be Harris recurrent.
(ii) If $C_+ = \mathsf{X}$, then the set C is said to be attractive.

If the domain of attraction C_+ of C contains C, then it may happen that $C_+ \subsetneq \mathsf{X}$. Nevertheless, as shown below, the set C_+ is absorbing.

2.7. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Let $C \in \mathscr{X}$ be a non-empty set such that $C \subset C_+$. Show that the set C_+ is absorbing.

Solutions to exercises

2.1 (i) Let $x \in C$. Since $\mathbb{P}_x(\sigma_C^{(n)} < \infty) = 1$ for all $x \in C$ and $n \in \mathbb{N}$, the strong Markov property applied to the Markov chain $\{X_n\}$ yields, for any $B \in \mathscr{X}$,

$$\mathbb{P}_x\left(\tilde{X}_{n+1} \in B \,\middle|\, \mathscr{F}_{\sigma_C^{(n)}}\right) = \mathbb{P}_x\left(X_{\sigma_C^{(n+1)}} \in B \,\middle|\, \mathscr{F}_{\sigma_C^{(n)}}\right) = \mathbb{P}_x\left(X_{\sigma_C} \circ \theta_{\sigma_C^{(n)}} \in B \,\middle|\, \mathscr{F}_{\sigma_C^{(n)}}\right)$$
$$= \mathbb{P}_{X_{\sigma_C^{(n)}}}(X_{\sigma_C} \in B) = Q_C(\tilde{X}_n, B) \,.$$

(ii) Since $A \subset C$, we have $\sigma_A = \sigma_C^{(\tilde{\sigma}_A)}$. Thus,

$$\sigma_A = \sum_{n=0}^{\tilde{\sigma}_A - 1} \{\sigma_C^{(n+1)} - \sigma_C^{(n)}\} = \sum_{n=0}^{\infty} \{\sigma_C^{(n+1)} - \sigma_C^{(n)}\} \mathbb{1}_{\{n < \tilde{\sigma}_A\}} = \sum_{n=0}^{\infty} \sigma_C \circ \theta_{\sigma_C^{(n)}} \mathbb{1}_{\{n < \tilde{\sigma}_A\}} \,.$$

Let $x \in C$. Note that $\{n < \tilde{\sigma}_A\} = \cap_{i=1}^{n} \{X_{\sigma^{(i)}} \notin A\} \in \mathscr{F}_{\sigma^{(n)}}$ and applying again **??**, we have $\mathbb{P}_x(\sigma_C^{(n)} < \infty) = 1$. We then obtain by the strong Markov property,

$$\mathbb{E}_x[\sigma_A] = \sum_{n=0}^{\infty} \mathbb{E}_x[\sigma_C \circ \theta_{\sigma_C^{(n)}} \mathbb{1}\{n < \tilde{\sigma}_A\}]$$
$$= \sum_{n=0}^{\infty} \mathbb{E}_x[\mathbb{1}\{n < \tilde{\sigma}_A\} \mathbb{E}_{X_{\sigma_C^{(n)}}}[\sigma_C]] \le \mathbb{E}_x[\tilde{\sigma}_A] \sup_{y \in C} \mathbb{E}_y[\sigma_C] \,.$$

2.2 By the strong Markov property, we get

$$U(x,A) = \mathbb{E}_x\left[\sum_{n=0}^{\infty} \mathbb{1}_A(X_n)\right] = \mathbb{E}_x\left[\sum_{n=\tau_A}^{\infty} \mathbb{1}_A(X_n) \mathbb{1}\{\tau_A < \infty\}\right]$$
$$= \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbb{1}_A(X_n \circ \theta_{\tau_A}) \mathbb{1}\{\tau_A < \infty\}\right]$$
$$= \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbb{1}\{\tau_A < \infty\} \mathbb{E}_{X_{\tau_A}}\left[\mathbb{1}_A(X_n)\right]\right] \le \mathbb{P}_x(\tau_A < \infty) \sup_{y \in A} U(y,A) \,.$$

2.3 Define $h(x) = \mathbb{P}_x(N_A = \infty)$. Then $Ph(x) = \mathbb{E}_x[h(X_1)] = \mathbb{E}_x[\mathbb{P}_{X_1}(N_A = \infty)]$ and applying the Markov property, we obtain

$$Ph(x) = \mathbb{E}_x[\mathbb{P}_x(N_A \circ \theta = \infty \,|\, \mathscr{F}_1)] = \mathbb{P}_x(N_A \circ \theta = \infty) = \mathbb{P}_x(N_A = \infty) = h(x) \,.$$

2.4 (i) For $p \in \mathbb{N}$, $\sigma_A^{(p+1)} = \sigma_A^{(p)} + \sigma_A \circ \theta_{\sigma_A^{(p)}}$ on $\{\sigma_A^{(p)} < \infty\}$. Applying the strong Markov property yields

$$\mathbb{P}_x(\sigma_A^{(p+1)} < \infty) = \mathbb{P}_x\left(\sigma_A^{(p)} < \infty, \sigma_A \circ \theta_{\sigma_A^{(p)}} < \infty\right)$$
$$= \mathbb{E}_x\left[\mathbb{1}\left\{\sigma_A^{(p)} < \infty\right\} \mathbb{P}_{X_{\sigma_A^{(p)}}}(\sigma_A < \infty)\right] \le \delta \mathbb{P}_x(\sigma_A^{(p)} < \infty) \,.$$

By induction, we obtain $\mathbb{P}_x(\sigma_A^{(p)} < \infty) \le \delta^p$ for every $p \in \mathbb{N}^*$ and $x \in A$. Thus, for $x \in A$,

$$U(x,A) = \mathbb{E}_x[N_A] \le 1 + \sum_{p=1}^{\infty} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \le (1-\delta)^{-1} \; .$$

Since by **??** for all $x \in \mathsf{X}$, $U(x,A) \le \sup_{y\in A} U(y,A)$, (2.4) follows.

(ii) By **??**, $\mathbb{P}_x(\sigma_A^{(n)} < \infty) = 1$ for every $n \in \mathbb{N}$ and $x \in A$. Then,

$$\mathbb{P}_x(N_A = \infty) = \mathbb{P}_x\left(\bigcap_{n=1}^{\infty}\{\sigma_A^{(n)} < \infty\}\right) = 1 \; .$$

2.5 1. Using the Markov property,

$$\begin{aligned}P^n f(x) = \mathbb{E}_x[f(X_n)] &= \mathbb{E}_x[\mathbb{1}\{n \le \sigma_A\} f(X_n)] + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{\sigma_A = j\} f(X_n)] \\ &= {}_A^n P f(x) + \sum_{j=1}^{n-1} \mathbb{E}_x\left[\mathbb{1}\{\sigma_A = j\}\mathbb{E}_{X_j}[f(X_{n-j})]\right] \\ &= {}_A^n P f(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{\sigma_A \ge j\}\mathbb{1}_A(X_j) P^{n-j} f(X_j)] \\ &= {}_A^n P f(x) + \sum_{j=1}^{n-1} {}_A^j P(\mathbb{1}_A \times P^{n-j} f)(x) \; . \end{aligned} \tag{2.9}$$

2. The last exit decomposition is established analogously.

$$\begin{aligned}P^n f(x) &= \mathbb{E}_x[f(X_n)] \\ &= \mathbb{E}_x[\mathbb{1}_{\{n\le\sigma_A\}} f(X_n)] + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{X_j \in A, X_{j+1} \notin A, \ldots, X_{n-1} \notin A\} f(X_n)] \\ &= {}_A^n P f(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}_A(X_j)\mathbb{E}_{X_j}[\mathbb{1}\{X_1 \notin A, \ldots, X_{n-j-1} \notin A\} f(X_{n-j})]] \\ &= {}_A^n P f(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}_A(X_j)\, {}_A^{n-j} P f(X_j)] \\ &= {}_A^n P f(x) + \sum_{j=1}^{n-1} P^j(\mathbb{1}_A \times {}_A^{n-j} P f)(x) \; . \end{aligned} \tag{2.10}$$

2.6 The assertion (iv) $\Rightarrow$ (i) is the only non trivial one. It means that if A can be reached from A^c, then it can be reached from A. Indeed, starting from A, either the chain remains in A, or it leaves A and then can reach it again. Formally, applying the Markov property yields

$$\begin{aligned}\mathbb{P}_x(\sigma_A < \infty) &= \mathbb{P}_x(X_1 \in A) + \mathbb{P}_x(X_1 \in A^c, \sigma_A \circ \theta < \infty) \\ &= \mathbb{P}_x(X_1 \in A) + \mathbb{E}_x[\mathbb{1}_{A^c}(X_1)\mathbb{P}_{X_1}(\sigma_A < \infty)] \; .\end{aligned}$$

For each $x \in \mathsf{X}$, either $\mathbb{P}_x(X_1 \in A) > 0$ or $\mathbb{P}_x(X_1 \in A) = 0$. In the latter case, it then holds that $\mathbb{P}_x(\sigma_A < \infty) = \mathbb{E}_x[\mathbb{1}_{A^c}(X_1)\mathbb{P}_{X_1}(\sigma_A < \infty)] > 0$ if (iv) holds. Thus (iv) $\Rightarrow$ (i).

2.7 Let $x \in C_+$. Then,

$$\begin{aligned} 0 = \mathbb{P}_x(\sigma_C = \infty) &\geq \mathbb{P}_x(X_1 \in C^c\,,\ \sigma_C \circ \theta = \infty) \\ &\geq \mathbb{P}_x(X_1 \in C^c_+\,,\ \sigma_C \circ \theta = \infty) = \mathbb{E}_x[\mathbb{1}_{C^c_+}(X_1)\mathbb{P}_{X_1}(\sigma_C = \infty)]\,. \end{aligned}$$

Since $\mathbb{P}_y(\sigma_C = \infty) > 0$ for $y \in C^c_+$, this yields $P(x, C^c_+) = \mathbb{P}_x(X_1 \in C^c_+) = 0$.