Chapter 1
Exercices Week 2

1.1 Exercises

1.1. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_k,\ k \in T\}, \mathbb{P})$ be a filtered probability space and $\{(X_k, \mathscr{F}_k),\ k \in T\}$ be an adapted stochastic process. Show that the following properties are equivalent.

(i) $\{(X_k, \mathscr{F}_k),\ k \in T\}$ is a Markov chain.
(ii) For every $k \in T$ and bounded $\sigma(X_j, j \geq k)$-measurable random variable Y,

$$\mathbb{E}\left[Y \mid \mathscr{F}_k\right] = \mathbb{E}\left[Y \mid X_k\right] \qquad \mathbb{P}-\text{a.s.} \tag{1.1}$$

(iii) For every $k \in T$, bounded $\sigma(X_j, j \geq k)$-measurable random variable Y and bounded $\mathscr{F}_k^X$-measurable random variable Z,

$$\mathbb{E}\left[YZ \mid X_k\right] = \mathbb{E}\left[Y \mid X_k\right]\mathbb{E}\left[Z \mid X_k\right] \qquad \mathbb{P}-\text{a.s.} \tag{1.2}$$

Definition 1.1 (Absorbing set) *A set $B \in \mathscr{X}$ is called absorbing if $P(x,B) = 1$ for all $x \in B$.*

This definition subsumes that the empty set is absorbing. Of course the interesting absorbing sets are non-empty.

1.2. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$ admitting an invariant probability measure π. If $B \in \mathscr{X}$ is an absorbing set, then $\pi_B = \pi(B \cap \cdot)$ is an invariant finite measure. Moreover, if the invariant probability measure is unique, then $\pi(B) \in \{0,1\}$.

Definition 1.2 *Let π_0, π_1 be two probability measure on $(\mathsf{X}, \mathscr{X})$. Then $\pi_0, \pi_1 \ll \lambda = \pi_0 + \pi_1$ and denoting $f_i = \mathrm{d}\pi_i / \mathrm{d}\lambda$ for $i = 0,1$, we define the two probability measures:*

$$(\pi_0 - \pi_1)_+(\mathsf{A}) = \int_{\mathsf{A}} (f_0 - f_1)_+(x) \mathrm{d}\lambda(x) \,, \quad (\pi_0 - \pi_1)_-(\mathsf{A}) = \int_{\mathsf{A}} (f_0 - f_1)_-(x) \mathrm{d}\lambda(x) \,. \tag{1.3}$$

1.3. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Show that

(i) The set of invariant probability measures for P is a convex subset of $\mathbb{M}_+(\mathscr{X})$.
(ii) For any two distinct invariant probability measures π, π' for P, the finite measures $(\pi - \pi')^+$ and $(\pi - \pi')^-$ are non-trivial, mutually singular and invariant for P.

Solutions to exercises

1.1 (i) $\Rightarrow$ (ii) Fix $k \in T$ and consider the property (where $\mathbb{F}_b(\mathsf{X})$ is the set of bounded measurable functions),

$(\mathscr{P}_n)$: (0.1) holds for all $Y = \prod_{j=0}^n g_j(X_{k+j})$ where $g_j \in \mathbb{F}_b(\mathsf{X})$ for all $j \geq 0$.

$(\mathscr{P}_0)$ is true. Assume that $(\mathscr{P}_n)$ holds and let $\{g_j,\ j \in \mathbb{N}\}$ be a sequence of functions in $\mathbb{F}_b(\mathsf{X})$. The Markov property (**??**) yields

$$\begin{aligned}
&\mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_k\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_{k+n}\right]\middle|\mathscr{F}_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})\mathbb{E}\left[g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_{k+n}\right]\middle|\mathscr{F}_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})\mathbb{E}\left[g_{n+1}(X_{k+n+1})\middle|X_{k+n}\right]\middle|\mathscr{F}_k\right] .
\end{aligned}$$

The last term in the product being a measurable function of X_{n+k}, the induction assumption $(\mathscr{P}_n)$ yields

$$\begin{aligned}
&\mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})\mathbb{E}\left[g_{n+1}(X_{k+n+1})\middle|X_{k+n}\right]\middle|X_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})\mathbb{E}\left[g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_{k+n}\right]\middle|X_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})g_{n+1}(X_{k+n+1})\middle|X_k\right] ,
\end{aligned}$$

which proves $(\mathscr{P}_{n+1})$. Therefore, $(\mathscr{P}_n)$ is true for all $n \in \mathbb{N}$.
Consider the set

$$\mathscr{H} = \left\{Y \in \sigma(X_j, j \geq k) : \mathbb{E}\left[Y\middle|\mathscr{F}_k\right] = \mathbb{E}\left[Y\middle|X_k\right] \ \ \mathbb{P}-\text{a.s.}\right\} .$$

It is easily seen that $\mathscr{H}$ is a vector space. In addition, if $\{Y_n,\ n \in \mathbb{N}\}$ is an increasing sequence of nonnegative random variables in $\mathscr{H}$ and if $Y = \lim_{n\to\infty} Y_n$ is bounded, then by the monotone convergence theorem for conditional expectations,

$$\mathbb{E}\left[Y\middle|\mathscr{F}_k\right] = \lim_{n\to\infty}\mathbb{E}\left[Y_n\middle|\mathscr{F}_k\right] = \lim_{n\to\infty}\mathbb{E}\left[Y_n\middle|X_k\right] = \mathbb{E}\left[Y\middle|X_k\right] \qquad \mathbb{P}-\text{a.s.}$$

By **??**, the space $\mathscr{H}$ contains all $\sigma(X_j, j \geq k)$ measurable random variables.

(ii) $\Rightarrow$ (iii) If Y is a bounded $\sigma(X_j, j \geq k)$-measurable random variable and Z is a bounded $\mathscr{F}_k$ measurable random variable, an application of (ii) yields

$$\mathbb{E}\left[YZ\middle|\mathscr{F}_k\right] = Z\mathbb{E}\left[Y\middle|\mathscr{F}_k\right] = Z\mathbb{E}\left[Y\middle|X_k\right] \qquad \mathbb{P}-\text{a.s.}$$

Thus,

$$\begin{aligned}
\mathbb{E}\left[YZ\middle|X_k\right] &= \mathbb{E}\left[\mathbb{E}\left[YZ\middle|\mathscr{F}_k\right]\middle|X_k\right] = \mathbb{E}\left[Z\mathbb{E}\left[Y\middle|X_k\right]\middle|X_k\right] \\
&= \mathbb{E}\left[Z\middle|X_k\right]\mathbb{E}\left[Y\middle|X_k\right] \qquad \mathbb{P}-\text{a.s.}
\end{aligned}$$

(iii) $\Rightarrow$ (i) If Z is bounded and $\mathscr{F}_k$-measurable, we obtain

$$\begin{aligned}\mathbb{E}\left[f(X_{k+1})Z\right] &= \mathbb{E}\left[\mathbb{E}\left[f(X_{k+1})Z|X_k\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f(X_{k+1})|X_k\right]\mathbb{E}\left[Z|X_k\right]\right] = \mathbb{E}\left[\mathbb{E}\left[f(X_{k+1})|X_k\right]Z\right] \ .\end{aligned}$$

This proves (i).