# Chapter 2
# Exercices Sheet 2

**2.1 (Discrete autoregressive process).** Consider the DAR(1) model defined by the recursion

$$X_k = V_k X_{k-1} + (1 - V_k) Z_k \;, \tag{2.1}$$

where $\{V_k,\ k \in \mathbb{N}\}$ is a sequence of i.i.d. Bernoulli random variables with $\mathbb{P}(V_k = 1) = \alpha \in [0,1)$, $\{Z_k,\ k \in \mathbb{N}\}$ are i.i.d. random variables with distribution $\pi$ on a measurable space $(\mathsf{X}, \mathscr{X})$ and $\{V_k,\ k \in \mathbb{N}\}$ and $\{Z_k,\ k \in \mathbb{N}\}$ are mutually independent and independent of $X_0$, whose distribution is $\xi$.

1. Show that $Pf(x) = \alpha f(x) + (1-\alpha)\pi(f)$.
2. Show that $\pi$ is the unique invariant probability.

Assume that $\mathsf{X} = \mathbb{N}$ and $\sum_{k=0}^{\infty} k^2 \pi(k) < \infty$ and that the distribution of $X_0$ is $\pi$.

3. Show that for any positive integer $h$, $\mathrm{Cov}(X_h, X_0) = \alpha^h \mathrm{Var}(X_0)$.

**2.2.** Consider a scalar AR(1) process $\{X_k,\ k \in \mathbb{N}\}$ defined recursively as follows:

$$X_k = \phi X_{k-1} + Z_k \;, \tag{2.2}$$

where $\{Z_k,\ k \in \mathbb{N}\}$ is a sequence of i.i.d. random variables, independent of $X_0$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\mathbb{E}[|Z_1|] < \infty$ and $\mathbb{E}[Z_1] = 0$.

1. Define the kernel $P$ of this chain.
2. Show that for all $k \geq 1$, $X_k$ has the same distribution as $\phi^k X_0 + B_k$ where $B_k = \sum_{j=0}^{k-1} \phi^j Z_j$.

Assume that $|\phi| < 1$.

3. Show that $B_k \stackrel{\mathbb{P}\text{-a.s.}}{\longrightarrow} B_\infty = \sum_{j=0}^{\infty} \phi^j Z_j$.
4. Show that the distribution of $B_\infty$ is the unique invariant probability of $P$.

Assume that $|\phi| > 1$ and the distribution of $\sum_{j=1}^{\infty} \phi^{-j} Z_j$ is continuous.

5. Show that for all $x \in \mathbb{R}$, $\mathbb{P}_x(\lim_{n\to\infty} |X_n| = \infty) = 1$.