Chapter 1
Exercices Sheet 1

1.1 Exercises

1.1. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_k,\ k \in T\}, \mathbb{P})$ be a filtered probability space and $\{(X_k, \mathscr{F}_k),\ k \in T\}$ be an adapted stochastic process. Show that the following properties are equivalent.

(i) $\{(X_k, \mathscr{F}_k),\ k \in T\}$ is a Markov chain.
(ii) For every $k \in T$ and bounded $\sigma(X_j, j \geq k)$-measurable random variable Y,

$$\mathbb{E}\left[Y \mid \mathscr{F}_k\right] = \mathbb{E}\left[Y \mid X_k\right] \qquad \mathbb{P}-\text{a.s.} \tag{1.1}$$

(iii) For every $k \in T$, bounded $\sigma(X_j, j \geq k)$-measurable random variable Y and bounded $\mathscr{F}_k^X$-measurable random variable Z,

$$\mathbb{E}\left[YZ \mid X_k\right] = \mathbb{E}\left[Y \mid X_k\right]\mathbb{E}\left[Z \mid X_k\right] \qquad \mathbb{P}-\text{a.s.} \tag{1.2}$$

1.2. Let $\{Z_n,\ n \in \mathbb{N}\}$ be an i.i.d. sequence of random variables independent of X_0. Define recursively $X_n = \phi X_{n-1} + Z_n$.

1. Show that $\{X_n,\ n \in \mathbb{N}\}$ defines a time-homogenous Markov chain.
2. Write its Markov kernel in the cases where (i) Z_1 is a Bernoulli random variable with probability of success $1/2$ and (ii) the law of Z_1 has a density q with respect to the Lebesgue measure.

1.3. Let a be a probability on $\mathbb{N}$, that is a sequence $\{a(n),\ n \in \mathbb{N}\}$ such that $a(n) \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{k=0}^{\infty} a(k) = 1$. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. The sampled kernel K_a is defined by

$$K_a = \sum_{n=0}^{\infty} a(n) P^n . \tag{1.3}$$

Let $\{a(n),\ n \in \mathbb{N}\}$ and $\{b(n),\ n \in \mathbb{N}\}$ be two sequences of real numbers. We denote by $\{a * b(n),\ n \in \mathbb{N}\}$ the convolution of the sequences a and b defined, for $n \in \mathbb{N}$ by

$$a * b(n) = \sum_{k=0}^{n} a(k)b(n-k) .$$

Show that if a and b are probabilities on $\mathbb{N}$, then the sampled kernels K_a and K_b satisfy the generalized Chapman-Kolmogorov equation

$$K_{a*b} = K_a K_b . \tag{1.4}$$

1.4. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$ and ν be a probability measure on $(\mathsf{X}, \mathscr{X})$. Show that an X-valued stochastic process $\{X_k,\ k \in \mathbb{N}\}$ is a homogeneous Markov chain with kernel P and initial distribution ν if and only if the distribution of $(X_0, \ldots, X_k)$ is $\nu \otimes P^{\otimes k}$ for all $k \in \mathbb{N}$.

Solutions to exercises

1.1 (i) $\Rightarrow$ (ii) Fix $k \in T$ and consider the property (where $\mathbb{F}_b(\mathsf{X})$ is the set of bounded measurable functions),

$(\mathscr{P}_n)$: (1.1) holds for all $Y = \prod_{j=0}^{n} g_j(X_{k+j})$ where $g_j \in \mathbb{F}_b(\mathsf{X})$ for all $j \geq 0$.

$(\mathscr{P}_0)$ is true. Assume that $(\mathscr{P}_n)$ holds and let $\{g_j,\ j \in \mathbb{N}\}$ be a sequence of functions in $\mathbb{F}_b(\mathsf{X})$. The Markov property yields

$$\begin{aligned}
&\mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_k\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_{k+n}\right]\middle|\mathscr{F}_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})\mathbb{E}\left[g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_{k+n}\right]\middle|\mathscr{F}_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})\mathbb{E}\left[g_{n+1}(X_{k+n+1})\middle|X_{k+n}\right]\middle|\mathscr{F}_k\right] .
\end{aligned}$$

The last term in the product being a measurable function of X_{n+k}, the induction assumption $(\mathscr{P}_n)$ yields

$$\begin{aligned}
&\mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})\mathbb{E}\left[g_{n+1}(X_{k+n+1})\middle|X_{k+n}\right]\middle|X_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})\mathbb{E}\left[g_{n+1}(X_{k+n+1})\middle|\mathscr{F}_{k+n}\right]\middle|X_k\right] \\
&= \mathbb{E}\left[g_0(X_k)\dots g_n(X_{k+n})g_{n+1}(X_{k+n+1})\middle|X_k\right] ,
\end{aligned}$$

which proves $(\mathscr{P}_{n+1})$. Therefore, $(\mathscr{P}_n)$ is true for all $n \in \mathbb{N}$.
Consider the set

$$\mathscr{H} = \left\{Y \in \sigma(X_j, j \geq k) : \mathbb{E}\left[Y\middle|\mathscr{F}_k\right] = \mathbb{E}\left[Y\middle|X_k\right] \ \ \mathbb{P}-\text{a.s.}\right\} .$$

It is easily seen that $\mathscr{H}$ is a vector space. In addition, if $\{Y_n,\ n \in \mathbb{N}\}$ is an increasing sequence of nonnegative random variables in $\mathscr{H}$ and if $Y = \lim_{n\to\infty} Y_n$ is bounded, then by the monotone convergence theorem for conditional expectations,

$$\mathbb{E}\left[Y\middle|\mathscr{F}_k\right] = \lim_{n\to\infty} \mathbb{E}\left[Y_n\middle|\mathscr{F}_k\right] = \lim_{n\to\infty} \mathbb{E}\left[Y_n\middle|X_k\right] = \mathbb{E}\left[Y\middle|X_k\right] \qquad \mathbb{P}-\text{a.s.}$$

By the monotone class theorem, the space $\mathscr{H}$ contains all $\sigma(X_j, j \geq k)$ measurable random variables.

(ii) $\Rightarrow$ (iii) If Y is a bounded $\sigma(X_j, j \geq k)$-measurable random variable and Z is a bounded $\mathscr{F}_k$ measurable random variable, an application of (ii) yields

$$\mathbb{E}\left[YZ\middle|\mathscr{F}_k\right] = Z\mathbb{E}\left[Y\middle|\mathscr{F}_k\right] = Z\mathbb{E}\left[Y\middle|X_k\right] \qquad \mathbb{P}-\text{a.s.}$$

Thus,

$$\begin{aligned}
\mathbb{E}\left[YZ\middle|X_k\right] &= \mathbb{E}\left[\mathbb{E}\left[YZ\middle|\mathscr{F}_k\right]\middle|X_k\right] = \mathbb{E}\left[Z\mathbb{E}\left[Y\middle|X_k\right]\middle|X_k\right] \\
&= \mathbb{E}\left[Z\middle|X_k\right]\mathbb{E}\left[Y\middle|X_k\right] \qquad \mathbb{P}-\text{a.s.}
\end{aligned}$$

(iii) $\Rightarrow$ (i) If Z is bounded and $\mathscr{F}_k$-measurable, we obtain

$$\begin{aligned}\mathbb{E}\left[f(X_{k+1})Z\right] &= \mathbb{E}\left[\mathbb{E}\left[f(X_{k+1})Z|X_k\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f(X_{k+1})|X_k\right]\mathbb{E}\left[Z|X_k\right]\right] = \mathbb{E}\left[\mathbb{E}\left[f(X_{k+1})|X_k\right]Z\right] \ .\end{aligned}$$

This proves (i).