$\boxed{X_n \overset{W}{\Rightarrow} X} \iff$ (a) or (b) or (c) or (d).

(a): $\forall h$ bounded and continuous, $\mathbb{E}(h(X_n)) \xrightarrow[n\to\infty]{} \mathbb{E}(h(X))$.

(b): $\forall$ set A s.t. $\mathbb{P}(X \in \partial A) = 0$, $\underbrace{\mathbb{E}(\mathbb{1}_A(X_n))}_{\mathbb{P}(X_n \in A)} \xrightarrow[n\to+\infty]{} \underbrace{\mathbb{E}(\mathbb{1}_A(X))}_{\mathbb{P}(X\in A)}$

(c) $\forall x \in \mathbb{R}$ s.t. $\mathbb{P}(X = x) = 0$, $\underbrace{\mathbb{E}(\mathbb{1}_{]-\infty,x]}(X_n))}_{\mathbb{P}(X_n \leq x)} \to \mathbb{E}(\mathbb{1}_{]-\infty,x]}(X))$. $\quad \mathbb{P}(X_n \leq x) \xrightarrow[n\to\infty]{} \mathbb{P}(X \leq x)$

cdf (cumulative distribution function).

(d) $\forall u \in \mathbb{R}$, $\mathbb{E}(\underbrace{e^{iuX_n}}_{h_u(X_n)}) \xrightarrow[n\to+\infty]{} \mathbb{E}(\underbrace{e^{iuX}}_{h_u(X)})$. (Characteristic function).

$\boxed{X_n \xrightarrow{\mathbb{P}\text{-prob}} X} \iff \forall \varepsilon > 0$, $\mathbb{P}(|X_n - X| \geq \varepsilon) \xrightarrow[n\to+\infty]{} 0$.

$\boxed{X_n \xrightarrow{\mathbb{P}\text{-a.s}} X}$. $\iff \mathbb{P}\left(\lim_{n\to+\infty} X_n = X\right) = 1$.

Prop:

If: $\begin{cases} X_n \overset{W}{\longrightarrow} X. \\ X_n \xrightarrow{\mathbb{P}\text{-prob}} X. \\ X_n \xrightarrow{\mathbb{P}\text{-a.s}} X \end{cases}$ then: $\forall f$ continuous, $\begin{cases} f(X_n) \overset{W}{\longrightarrow} f(X). \\ f(X_n) \xrightarrow{\mathbb{P}\text{-prob}} f(X) \\ f(X_n) \xrightarrow{\mathbb{P}\text{-a.s}} f(X). \end{cases}$

We have: $\boxed{X_n \xrightarrow{\mathbb{P}\text{-a.s}} X \Rightarrow X_n \xrightarrow{\mathbb{P}\text{-prob}} X \Rightarrow X_n \overset{W.}{\Rightarrow} X.}$

Chap 3.

Strong Law of Large Numbers:

If : $\begin{cases} (1)\ (X_i)_{i\geq 1} \text{ are i.i.d (independent and identically distributed)} \\ (2)\ \mathbb{E}(|X_1|) < \infty. \end{cases}$

then: $\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mathbb{E}(X_1)$. $\mathbb{P}$-a.s.

Central limit theorem.

If : $\begin{cases} (1)\ (X_i)_{i\geq 1} \text{ are i.i.d.} \\ (2)\ \mathbb{E}(X_1^2) < \infty. \end{cases}$

then:

$$Z_n = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_1)}{\sqrt{\frac{Var(X_1)}{n}}} \xrightarrow{\mathcal{L}} Z \quad \text{where } Z \sim \mathcal{N}(0,1).$$

Equivalently : $\tilde{Z}_n = \sqrt{n}\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\overline{X}_n} - \mathbb{E}(X_1)\right) \xrightarrow{\mathcal{L}} \tilde{Z}$ where $\tilde{Z} \sim \mathcal{N}(0, Var(X_1))$.

$$\mathbb{E}(Z_n) = \frac{1}{\sqrt{\frac{Var(X_1)}{n}}} \underbrace{\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_1)\right)}_{\underbrace{\frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}(X_i)}_{\mathbb{E}(X_1)} - \mathbb{E}(X_1)}_{\underbrace{\mathbb{E}(X_1) - \mathbb{E}(X_1)}_{=0.}}} = 0$$

$$Var(Z_n) = \frac{1}{\frac{Var(X_1)}{n}} \underbrace{Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_1)\right)}_{\boxed{\underbrace{Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}_{\frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)}}}$$

because (X_i) are independent

$$= \frac{1}{n^2}\sum_{i=1}^{n} \overbrace{Var(X_i)}^{Var(X_1).} = \frac{1}{n^2}\, n.Var(X_1)$$

$$= \frac{1}{n}\operatorname{Var}(X_1)$$

$$= \frac{1}{\frac{\operatorname{Var}(X_1)}{n}} \cdot \frac{\operatorname{Var}(X_1)}{n} = 1.$$

<u>Remark:</u> $Z_n \xrightarrow{\mathcal{L}} Z \Rightarrow \tilde{Z}_n \xrightarrow{\mathcal{L}} \tilde{Z}$.

<u>Indeed:</u> $\left\{\begin{array}{l} Z_n \xrightarrow{\mathcal{L}} Z \\ f(z) = z \cdot \sqrt{\operatorname{Var}(X_1)} \\ \text{continuous} \end{array}\right. \Rightarrow \left\{\begin{array}{l} \tilde{Z}_n = f(Z_n) \\ \quad = Z_n \times \sqrt{\operatorname{Var}(X_1)} \end{array}\right. \xrightarrow{\mathcal{L}} \underbrace{f(Z)}_{\tilde{Z}} = Z\sqrt{\operatorname{Var}(X_1)}$

Since: $Z \sim \mathcal{N}(0,1)$, $\tilde{Z} = Z\sqrt{\operatorname{Var}(X_1)} \sim \mathcal{N}(\underset{\mathbb{E}(\tilde{Z})}{\underset{\uparrow}{0}}, \underbrace{\operatorname{Var}(Z)}_{1} \cdot \operatorname{Var}(X_1))$

$$\tilde{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mathbb{E}(X))^2, \quad \underbrace{\hat{\sigma}_N^2}_{} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \overline{X}_N)^2.$$

Let us show: $\mathbb{E}[\tilde{\sigma}_N^2] = \mathbb{E}(\hat{\sigma}_N^2) = \underbrace{\sigma^2}_{}$ (where: $\underbrace{\sigma^2}_{} = \underbrace{\operatorname{Var}(X_1)}_{}$).

$$\tilde{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mathbb{E}(X_1))^2. \qquad \text{Set: } \tilde{X}_i = \underbrace{X_i - \mathbb{E}(X_1)}_{}.$$

$$\tilde{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^{N}\tilde{X}_i^2$$

$$\mathbb{E}[\tilde{\sigma}_N^2] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\tilde{X}_i^2\right] = \frac{1}{N}\mathbb{E}\left[\sum_{i=1}^{N}\tilde{X}_i^2\right].$$

$$= \frac{1}{N}\sum_{i=1}^{N}\underbrace{\mathbb{E}[\tilde{X}_i^2]}_{\mathbb{E}[\tilde{X}_1^2]}.$$

$$= \frac{\cancel{N}}{\cancel{N}}\mathbb{E}[\tilde{X}_1^2].$$

$$= \mathbb{E}[\tilde{X}_1^2] = \mathbb{E}\left[(X_1 - \mathbb{E}(X_1))^2\right] = \operatorname{Var}(X_1)$$

$$= \sigma^2$$

$$\boxed{\hat{\sigma}_N^2} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X}_N)^2 \quad . \text{ Setting } \tilde{X}_i = X_i - E(X_1) \qquad \bar{X}_N = \frac{1}{N}\sum_{j=1}^{N} X_j .$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\left(\tilde{X}_i + \cancel{E(X_1)} - \left(\frac{1}{N}\sum_{j=1}^{N}\tilde{X}_j + \cancel{E(X_1)}\right)\right)^2 \qquad \bar{\tilde{X}}_N = \underbrace{\frac{1}{N}\sum_{i=1}^{N}\tilde{X}_i}$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\left(\tilde{X}_i - \bar{\tilde{X}}_N\right)^2$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\left(\tilde{X}_i^2 + \bar{\tilde{X}}_N^2 - 2\tilde{X}_i \cdot \bar{\tilde{X}}_N\right)$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\tilde{X}_i^2 + \frac{N}{N-1}\bar{\tilde{X}}_N^2 - \frac{2}{N-1}\underbrace{\sum_{i=1}^{N}\tilde{X}_i}_{N\cdot\bar{\tilde{X}}_N} \cdot \bar{\tilde{X}}_N$$

$$\hat{\sigma}_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}\tilde{X}_i^2 - \frac{N}{N-1}\bar{\tilde{X}}_N^2$$

$$\boxed{\mathbb{E}(\hat{\sigma}_N^2)} = \mathbb{E}\left[\frac{1}{N-1}\sum_{i=1}^{N}\tilde{X}_i^2 - \frac{N}{N-1}\bar{\tilde{X}}_N^2\right]$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\underbrace{\mathbb{E}(\tilde{X}_i^2)}_{\underbrace{\mathbb{E}(\tilde{X}_1^2)}_{\underbrace{\mathbb{E}\left[(X_1 - \mathbb{E}(X_1))^2\right]}_{\sigma^2}}} - \frac{N}{N-1}\underbrace{\mathbb{E}(\bar{\tilde{X}}^2)}_{\underbrace{\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}(X_i - \mathbb{E}(X_1))\right)^2\right]}_{\underbrace{\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N}X_i\right)}_{= \frac{\mathrm{Var}(X_1)}{N} = \frac{\sigma^2}{N}.}}}$$

$$= \frac{N}{N-1}\sigma^2 - \frac{\cancel{N}}{N-1}\cdot\frac{\sigma^2}{\cancel{N}}.$$

$$= \frac{N-1}{N-1}\sigma^2 = \boxed{\sigma^2.}$$

Recall that $\tilde{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - E(X_1))^2$.

By the Law of Large Numbers, $\frac{1}{N}\sum_{i=1}^{N}(X_i - E(X_1))^2 \xrightarrow[N\to+\infty]{a.s.} \underbrace{\mathbb{E}\left((X_1 - \mathbb{E}(X_1))^2\right)}_{\sigma^2.}$

$$\hat{\sigma}_N^2 = \frac{1}{N-1}\underbrace{\sum_{i=1}^{N}\tilde{X}_i^2}_{,} - \frac{N}{N-1}\underbrace{\tilde{\bar{X}}_N^2}_{\left(\frac{1}{N}\sum_{i=1}^{N}X_i - \mathbb{E}(X_1)\right)^2}$$

$$= \underbrace{\frac{N}{N-1}}_{\xrightarrow[N\to+\infty]{} 1}\left(\underbrace{\frac{1}{N}\sum_{i=1}^{N}(X_i-\mathbb{E}(X_1))^2}_{\xrightarrow{a.s}\ \sigma^2\ (LLN).}\right) - \underbrace{\frac{N}{N-1}}_{\xrightarrow[N\to+\infty]{} 1.}\left(\underbrace{\frac{1}{N}\sum_{i=1}^{N}X_i-\mathbb{E}(X_1)}_{\xrightarrow[N\to+\infty]{} 0.}\right)^2$$

$$\xrightarrow[N\to+\infty]{a.s} 1\times\sigma^2 - 1\times 0 = \sigma^2.$$

An important tool: Slutsky's Lemma:

$$\left[\ \text{if}\ \left|\begin{array}{l} X_n \xrightarrow{\mathcal{L}} X. \\ (Y_n \xrightarrow{\mathbb{P}\text{-prob}} a) \iff (Y_n \xrightarrow{\mathcal{L}} a) \end{array}\right.\right. \quad \text{then } \forall f \text{ continuous, } f(X_n, Y_n) \xrightarrow{\mathcal{L}} f(X, a).$$

<u>Remark:</u> $Y_n \xrightarrow{\mathbb{P}\text{-prob}} Y \Rightarrow Y_n \xrightarrow{\mathcal{L}} Y.$

If Y is a constant called "a" then: $Y_n \xrightarrow{\mathcal{L}} a \Rightarrow Y_n \xrightarrow{\mathbb{P}\text{-prob.}} a.$

Let us show: $\dfrac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{\frac{\tilde{\sigma}_N^2}{N}}} \xrightarrow{\mathcal{L}} Z$ where $Z \sim \mathcal{N}(0,1)$

By the Central Limit Theorem, $Z_n = \dfrac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{\frac{\sigma^2}{n}}} \xrightarrow{\mathcal{L}} Z$ where $Z \sim \mathcal{N}(0,1)$

Moreover: $U_n = \sqrt{\dfrac{\sigma^2}{\tilde{\sigma}_N^2}} \xrightarrow[n\to+\infty]{a.s} 1$ because: $\tilde{\sigma}_N^2 \to \sigma^2$ ($\tilde{\sigma}_N^2$ strongly convergent).

So: $U_n \xrightarrow{\mathbb{P}\text{-prob}} 1 = u.$

Set $f(z,u) = z \times u$, f is continuous

By Slutsky's Lemma,

$$\frac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{\frac{\tilde{\sigma}_N^2}{N}}} = \overbrace{\frac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{\frac{\sigma^2}{n}}}}^{Z_n} \times \overbrace{\sqrt{\frac{\sigma^2}{\tilde{\sigma}_N^2}}}^{U_n} = f(Z_n, U_n) \xrightarrow{\mathcal{L}} \underbrace{f(Z, \underset{=1}{U})}_{Z \times 1 = Z}.$$

· δ-method

$$\text{If: } \left\{ \begin{array}{l} \overbrace{\sqrt{n}(X_n - a)}^{Z_n} \xrightarrow{\mathcal{L}} Z \\ g \text{ differentiable at } a \end{array} \right\} \Rightarrow \underbrace{\sqrt{n}\,(g(X_n) - g(a))}_{\tilde{Z}_n} \xrightarrow{\mathcal{L}} g'(a) Z.$$

$$\tilde{Z}_n = \sqrt{n}\,(g(X_n) - g(a)) = \underbrace{\sqrt{n}\,(X_n - a)}_{Z_n} \times \underbrace{\frac{g(X_n) - g(a)}{X_n - a}}_{U_n} = f(Z_n, U_n)$$

· Now: if $\boxed{U_n \xrightarrow{\mathbb{P}\text{-prob}} g'(a)}$ (*)

then, since: $Z_n \xrightarrow{\mathcal{L}} Z$, we have by Slutsky's Lemma:

$$f(Z_n, U_n) = Z_n \times U_n = \tilde{Z}_n \xrightarrow{\mathcal{L}} f(Z, g'(a)) = g'(a) Z.$$

To conclude, it remains to prove: $U_n \xrightarrow{\mathbb{P}\text{-prob}} g'(a)$.

$$\text{if } \left\{ \begin{array}{l} \boxed{X_n \xrightarrow{\mathbb{P}\text{-prob}} a.} \quad (**) \\ x \overset{\varphi}{\mapsto} \left\{ \begin{array}{ll} \frac{g(x) - g(a)}{x - a} & x \neq a. \\ g'(a) & x = a. \end{array} \right. \quad \Big| \ \varphi \text{ continuous} \end{array} \right] \Rightarrow \begin{array}{l} \varphi(X_n) \xrightarrow{\mathbb{P}\text{-prob}} \varphi(a). \\ \underbrace{\frac{g(X_n) - g(a)}{X_n - a}}_{U_n} \xrightarrow{\mathbb{P}\text{-prob}} g'(a) \end{array}$$

Let us show (**), $\left(\begin{array}{l} Z_n = \sqrt{n}(X_n - a) \xrightarrow{\mathcal{L}} Z. \\ U_n = \frac{1}{\sqrt{n}} \xrightarrow{\mathbb{P}\text{-prob}} 0. \end{array} \right.$

$$X_n = Z_n \times \frac{1}{\sqrt{n}} + a = Z_n \times U_n + a := \Psi(Z_n, U_n)$$

where : $\Psi(z,u) = z \times u + a$. is continuous.

By Slutsky's lemma: $\underbrace{\Psi(Z_n, U_n)}_{Z_n U_n + a = X_n} \xrightarrow{\mathcal{L}} \underbrace{\Psi(Z, 0)}_{Z \times 0 + a = a}$.

$$Z_n U_n + a = X_n \xrightarrow{\mathcal{L}} Z \times 0 + a = a.$$

Then $X_n \xrightarrow{\mathcal{L}} a$ which is equivalent to: $\boxed{X_n \xrightarrow{\mathbb{P}\text{-prob}} a.}$