MCMC Exam. Answers

23 October

1. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded, measurable function and $A \in \mathcal{B}(\mathbb{R})$.

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_A(X_k)h(X_{k+1})] &= \mathbb{E}[\mathbb{E}[h(X_{k+1})\mathbb{1}_A(X_k)|X_k]] \\
&= \frac{1}{\sqrt{2\pi}\sigma}\mathbb{E}[\mathbb{1}_A(X_k)\int_z h(X_k + |z|)e^{-\frac{z^2}{2\sigma^2}}\,\mathrm{d}z] \\
&= \frac{1}{\sqrt{2\pi}\sigma}\mathbb{E}[\mathbb{1}_A(X_k)\left(\int_{z>0} h(X_k + z)e^{\frac{-z^2}{2\sigma^2}}\,\mathrm{d}z + \int_{z\leqslant 0} h(X_k - z)e^{\frac{z^2}{2\sigma^2}}\,\mathrm{d}z\right)] \\
&= 2\frac{1}{\sqrt{2\pi}\sigma}\mathbb{E}[\mathbb{1}_A(X_k)\int_{z>0} h(X_k + z)e^{\frac{-z^2}{2\sigma^2}}\,\mathrm{d}z] \\
&= 2\frac{1}{\sqrt{2\pi}\sigma}\mathbb{E}[\mathbb{1}_A(X_k) int_{v>X_k} h(v)e^{\frac{-(v-X_k)^2}{2\sigma^2}}\,\mathrm{d}v] \\
&= 2\frac{1}{\sqrt{2\pi}\sigma}\mathbb{E}[\mathbb{1}_A\int_{v\in\mathbb{R}} h(v)\mathbb{1}_{v\geqslant X_k}e^{\frac{-(v-X_k)^2}{2\sigma^2}}\,\mathrm{d}v]
\end{aligned}$$

   Therefore, $Q(x, \mathrm{d}y) = 2\mathbb{1}_{y\geqslant x}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-x)^2}{2\sigma^2}}\,\mathrm{d}y$.

2. We always have that $X_1 \geqslant X_0$. Thus, the condition is $b_2 < a_1$.

3. No, it is not. Because for I_1, I_2 as in the previous answer we will always have $\mathbb{P}_\nu(X_0 \in I_2, X_1 \in I_1) > 0$ and $\mathbb{P}_\nu(X_0 \in I_1, X_1 \in I_2) = 0$.

4. By law of large numbers $\frac{X_k}{k} = \frac{X_0+\sum_{i=1}^k |\eta_i|}{k} \to \mathbb{E}[|\eta_1|]$. Therefore, $X_k \to +\infty$.

5. $P(x, p_1, y, p_2) = \alpha(x, y)Q(x, \mathrm{d}y)\delta_{p_1}(p_2) + (1 - \bar{\alpha}(x))\delta_x(\mathrm{d}y)\delta_{-p_1}(p_2)$, where $\alpha(x, y) = \frac{\pi(y)}{\pi(x)} \wedge 1$ and $\bar{\alpha}(x) = \int_{y\in\mathbb{R}} \alpha(x, y)Q(x, \mathrm{d}y)$ and Q the kernel from question 1.

6. S is the kernel of Metropolis-Hastings with proposal $Y_{k+1}, \hat{p}_{k+1} \sim Q(X_k, \mathrm{d}y)\delta_{-p_k}(\mathrm{d}p)$ and acceptance probability $\alpha(x, y, p_1, p_2) = \frac{\pi(y)}{\pi(x)} \wedge 1$.

7. S is Π reversible as a step of the Metropolis-Hastings algorithm. To show that R is reversible, we take $h : \mathsf{X}^2 \to \mathbb{R}$ a bounded measurable function and we notice that

$$\int h(x, l_1, y, l_2)\Pi(\mathrm{d}x\,\mathrm{d}l_1)R(x, l_1, \mathrm{d}y, \mathrm{d}l_2) = \int h(x, l_1, x, -l_1)\pi(\mathrm{d}x)\delta_{-1,1}(\mathrm{d}l_1) = \int h(x, l_1, y, l_2)\Pi(\mathrm{d}y\,\mathrm{d}l_2)R(y, l_2, \mathrm{d}x, \mathrm{d}l_1)$$

8. Π is invariant by RQ since it is invariant by both R and S. However, it is not reversible.

$$\mathbb{P}_\Pi(X_0 > 0, p_0 = 1, X_1 < 0, p_1 = 1) = 0 \neq \mathbb{P}_\Pi(X_0 < 0, p_0 = 1, X_0 > 0, p_1 = 1)$$

9. We have

$$\begin{aligned}
P(x, 1, A, \{1\}) &= \mathbb{P}(X_{k+1} \in A, p_{k+1} = 1|X_k = x, p_k = 1) \\
&= \mathbb{P}(x + |Z_{k+1}| \in A, p_{k+1} = 1|X_k = x, p_k = 1)\,.
\end{aligned}$$

   This probability is non-zero as soon as the Lebesgue measure of $A \cap [x, +\infty)$ is non-zero. Similarly, this probability is zero, as soon as the Lebesgue measure of $A \cap [x, +\infty)$ is zero.

10. For any A of positive Lebesgue measure and $i \in \{-1, 1\}$, the state $A \times \{i\}$ will be attained with positive probability in at most three steps.

Indeed, assume that $(X_0, p_0) = (x, 1)$. Then, either $\lambda(A \cap [x, +\infty)) > 0$ or $\lambda(A \cap) - \infty, x]) > 0$.

In the first case, we have $\mathbb{P}(X_1 \in A, p_1 = 1 | (X_0, p_0) = (x, 1)) > 0$ (we do not flip p_0) and $\mathbb{P}(X_2 \in A, p_2 = -1 | (X_0, p_0) = (x, 1)) > 0$ (we flip p_1). Thus, the state $A \times \{i\}$ is attained in at most 2 steps.

In the case where $\lambda(A \cap) - \infty, x]) > 0$, we have

$\mathbb{P}(X_1 = x, p_1 = -1 | (X_0, p_0) = (x, 1)) > 0$ and thus $\mathbb{P}(X_2 \in A, p_2 = -1 | (X_0, p_0) = (x, 1)) > 0$ and $\mathbb{P}(X_3 \in A, p_3 = 1 | (X_0, p_0) = (x, 1)) > 0$. Thus, the state $A \times \{i\}$ is attained in at most 3 steps.

The case, where the initial state is $(x, -1)$ is done analogously.

As a conclusion, denoting λ the Lebesgue measure on $\mathbb{R}$ and defining $\nu(A \times \{i\}) = \lambda(A)$, we find that P is ν-irreducible. This implies that Π is the unique invariant measure of P.

Therefore, for any (X_k, p_k) produced by the algorithm, and any $h : \mathbb{R} \to \mathbb{R}$, a bounded measurable function,

$$\frac{1}{N} \sum_{i=1}^{n} h(X_k) \to n \to +\infty \mathbb{E}_{X,p\sim\Pi} \mathbb{E}[h(X)] = \mathbb{E}_{X\sim\pi} \mathbb{E}[h(X)] \, ,$$

where the last equality comes from the fact that if $(X, p) \sim \Pi$, then $X \sim \pi$.