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Stochastic calculus for machine learning, Master MVA

2 Brownian motion

If not specified, $(B_t)_{t\geq 0}$ is a standard Brownian motion.

Exercise 2.1. Let $\{(B_t^n)_{t\geq 0} : n \in \mathbb{N}\}$ be a sequence of independent Brownian motions. Show that the process $(W_t)_{t\geq 0}$ defined as

$$W_t = B_{t-\lfloor t\rfloor}^{\lfloor t\rfloor} + \sum_{i=0}^{\lfloor t\rfloor -1} B_1^i \,, \tag{9}$$

is a Brownian motion. Note here we use the convention that $\sum_{i=0}^{-1} = 0$.

Exercise 2.2. Let $T = \inf\{t \geq 0 : B_t = 1\}$ with the convention $\inf \varnothing = +\infty$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Show that $\mathbb{P}(T < \infty) \geq 1/2$.

Exercise 2.3. What is the distribution of $G = \int_0^1 B_t \, \mathrm{d}t$.

Exercise 2.4. Let $G = \int_0^2 B_t \, \mathrm{d}t$. Compute the conditional expectation $\mathbb{E}[B_1 \mid G]$.

Exercise 2.5. Show that the integral

$$\int_0^1 \frac{B_s}{s} \, \mathrm{d}s$$

is well defined almost surely for a well-chosen standard Brownian motion $(B_s)_{s\geq 0}$.

Exercise 2.6. Let $\beta_t = B_t - \int_0^t \frac{B_s}{s} \, \mathrm{d}s$. Show that $(\beta_t)_{t\geq 0}$ is a Brownian motion.

Exercise 2.7. Show that $\int_0^\infty |B_s| \, \mathrm{d}s = +\infty$ almost surely.

Exercise 2.8. For each $t \in [0,1]$, define

$$\mathcal{F}_t = \sigma(B_s, \; s \in [0,t]), \qquad \mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_1).$$

(1) Let $0 \leq s < t \leq 1$. Show that

$$\mathbb{E}[B_t - B_s \mid \mathcal{G}_s] = \frac{t-s}{1-s} (B_1 - B_s) \,.$$

Hint: Show that $\mathbb{E}[B_t - B_s \mid \mathcal{G}_s] = \mathbb{E}[B_t - B_s \mid B_1 - B_s]$ by Proposition 1.1 and conclude by Proposition 1.9.

(2) Consider the process $\{\beta_t : t \in [0,1]\}$ defined by

$$\beta_t = B_t - \int_0^t \frac{B_1 - B_s}{1-s} \, \mathrm{d}s \,, \qquad t \in [0,1] \,.$$

Show that for $0 \leq s < t \leq 1$,

$$\mathbb{E}[\beta_t \mid \mathcal{G}_s] = \beta_s \quad \text{almost surely} \,.$$