Chapter 2
Exercices Week 2

2.1 (Maximum principle). Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Show that for all $x \in \mathsf{X}$ and $A \in \mathscr{X}$,

$$U(x,A) \leq \mathbb{P}_x(\tau_A < \infty) \sup_{y \in A} U(y,A) .$$

2.2. Show that for every $A \in \mathscr{X}$, the function $x \mapsto \mathbb{P}_x(N_A = \infty)$ is harmonic.

2.3 (The Kac Formula). Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$ with invariant probability measure π. For all $C \in \mathscr{X}$ such that $\mathbb{P}_x(\sigma_C < \infty) > 0$ for π-almost all $x \in \mathsf{X}$, define

$$\pi_C^0(f) = \int_C \pi(\mathrm{d}x) \mathbb{E}_x \left[\sum_{k=0}^{\sigma_C - 1} f(X_k) \right] , \quad \pi_C^1(f) = \int_C \pi(\mathrm{d}x) \mathbb{E}_x \left[\sum_{k=1}^{\sigma_C} f(X_k) \right]$$

1. Show that for all $f \in \mathbb{F}_+(\mathsf{X})$,

$$\pi(f) = \sum_{\ell=1}^{n} \mathbb{E}_\pi \left[\mathbb{1}_C(X_\ell) \mathbb{E}_{X_\ell} \left[f(X_{n-\ell}) \mathbb{1}_{C^c}(X_1) \ldots \mathbb{1}_{C^c}(X_{n-\ell}) \right] \right] + \mathbb{E}_\pi[f(X_n) \mathbb{1}\{\sigma_C > n\}]$$

2. Show that

$$\pi(f) = \pi(f) = \pi_C^0(f) + \lim_{n \to \infty} \mathbb{E}_\pi[f(X_n) \mathbb{1}\{\sigma_C > n\}]$$

3. Set for $x \in \mathsf{X}$, $g(x) = \mathbb{P}_x(\sigma_C < \infty)$. Show that $\lim_{n \to \infty} \mathbb{E}_\pi[g(X_n) \mathbb{1}\{\sigma_C > n\}] = 0$.
4. Denote $\Delta\pi = \pi - \pi^0$. Show that $\Delta\pi \ll \pi$.
5. Show that

$$\pi = \pi_C^0 = \pi_C^1$$

2.4. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$ with invariant probability measure π. If $\mathbb{P}_x(\sigma_C < \infty) > 0$ for π-almost all $x \in \mathsf{X}$, then $\mathbb{P}_\pi(\sigma_C < \infty) = 1$.

2.5. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$ with invariant probability measure π. Assume that there exists non-negative functions V and f and a constant d such that

$$PV + f \leq V + d$$

Show that $\pi(f) < \infty$.

2.6. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Let $A \in \mathscr{X}$.

(i) Assume that there exists $\delta \in [0,1)$ such that $\sup_{x\in A} \mathbb{P}_x(\sigma_A < \infty) \leq \delta$. Show that for all $p \in \mathbb{N}^*$, $\sup_{x\in A} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \leq \delta^p$ and $\sup_{x\in \mathsf{X}} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \leq \delta^{p-1}$. Moreover,

$$\sup_{x\in \mathsf{X}} U(x,A) \leq (1-\delta)^{-1} . \tag{2.1}$$

(ii) Assume that $\mathbb{P}_x(\sigma_A < \infty) = 1$ for all $x \in A$. Show that for all $p \in \mathbb{N}^*$, $\inf_{x\in A} \mathbb{P}_x(\sigma_A^{(p)} < \infty) = 1$. Moreover, $\inf_{x\in A} \mathbb{P}_x(N_A = \infty) = 1$ for all $x \in A$.

Given $A \in \mathscr{X}$, we define, for $n \geq 1$ and $B \in \mathscr{X}$,

$$ {}_A^nP(x,B) = \mathbb{P}_x(X_n \in B\,,\, n \leq \sigma_A) . \tag{2.2}$$

Thus ${}_A^nP(x,B)$ is the probability that the chain goes from x to B in n steps without visiting the set A. It is called the n-step taboo probability. Note that ${}_A^1P = P$ and ${}_A^nP = (PI_{A^c})^{n-1}P$ where I_A is the kernel defined by $I_A f(x) = \mathbb{1}_A(x) f(x)$ for any $f \in \mathbb{F}_+(\mathsf{X})$

2.7. 1. Show the *first-entrance decomposition*

$$P^n f(x) = {}_A^nPf(x) + \sum_{j=1}^{n-1} {}_A^jP(\mathbb{1}_A \times P^{n-j} f)(x) . \tag{2.3}$$

2. Show the *last exit decomposition*

$$P^n f(x) = {}_A^nPf(x) + \sum_{j=1}^{n-1} P^j(\mathbb{1}_A \times {}_A^{n-j}Pf)(x) . \tag{2.4}$$

Solutions to exercises

2.1 By the strong Markov property, we get

$$\begin{aligned}U(x,A) &= \mathbb{E}_x\left[\sum_{n=0}^{\infty} \mathbb{1}_A(X_n)\right] = \mathbb{E}_x\left[\sum_{n=\tau_A}^{\infty} \mathbb{1}_A(X_n)\mathbb{1}\{\tau_A<\infty\}\right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbb{1}_A(X_n\circ\theta_{\tau_A})\mathbb{1}\{\tau_A<\infty\}\right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbb{1}\{\tau_A<\infty\}\mathbb{E}_{X_{\tau_A}}\left[\mathbb{1}_A(X_n)\right]\right] \le \mathbb{P}_x(\tau_A<\infty)\sup_{y\in A} U(y,A)\ .\end{aligned}$$

2.2 Define $h(x)=\mathbb{P}_x(N_A=\infty)$. Then $Ph(x)=\mathbb{E}_x[h(X_1)]=\mathbb{E}_x[\mathbb{P}_{X_1}(N_A=\infty)]$ and applying the Markov property, we obtain

$$Ph(x)=\mathbb{E}_x[\mathbb{P}_x(N_A\circ\theta=\infty\,|\,\mathscr{F}_1)]=\mathbb{P}_x(N_A\circ\theta=\infty)=\mathbb{P}_x(N_A=\infty)=h(x)\ .$$

2.3 1. By the last-exit decomposition, we have for all measurable nonnegative functions f,

$$\begin{aligned}\pi(f) = \mathbb{E}_\pi[f(X_n)] &= \sum_{\ell=1}^{n} \mathbb{E}_\pi\left[f(X_n)\mathbb{1}_C(X_\ell)\mathbb{1}_{C^c}(X_{\ell+1})\dots\mathbb{1}_{C^c}(X_n)\right]+\mathbb{E}_\pi[f(X_n)\mathbb{1}\{\sigma_C>n\}] \\ &= \sum_{\ell=1}^{n} \mathbb{E}_\pi\left[\mathbb{1}_C(X_\ell)\mathbb{E}_{X_\ell}\left[f(X_{n-\ell})\mathbb{1}_{C^c}(X_1)\dots\mathbb{1}_{C^c}(X_{n-\ell})\right]\right]+\mathbb{E}_\pi[f(X_n)\mathbb{1}\{\sigma_C>n\}]\end{aligned}$$

2. Noting that π is invariant and setting $k=n-\ell$, we get

$$\begin{aligned}\pi(f) &= \sum_{k=0}^{n-1}\int_C \pi(\mathrm{d}x)\mathbb{E}_x[f(X_k)\mathbb{1}\{\sigma_C>k\}]+\mathbb{E}_\pi[f(X_n)\mathbb{1}\{\sigma_C>n\}] \\ &= \int_C \pi(\mathrm{d}x)\mathbb{E}_x\left[\sum_{k=0}^{(n-1)\wedge(\sigma_C-1)} f(X_k)\right]+\mathbb{E}_\pi[f(X_n)\mathbb{1}\{\sigma_C>n\}]\end{aligned}$$

Thus, $\pi(f)=\pi^0_C(f)+\lim_{n\to\infty}\mathbb{E}_\pi[f(X_n)\mathbb{1}\{\sigma_C>n\}]$.

3. We get

$$\begin{aligned}\mathbb{E}_\pi[g(X_n)\mathbb{1}\{\sigma_C>n\}] &= \mathbb{E}_\pi[\mathbb{P}_{X_n}(\sigma_C<\infty)\mathbb{1}\{\sigma_C>n\}] \\ &= \mathbb{E}_\pi[\mathbb{1}\{\sigma_C\circ\theta^n<\infty\}\mathbb{1}\{\sigma_C>n\}] \\ &= \mathbb{E}_\pi[\mathbb{1}\{\sigma_C<\infty\}\mathbb{1}\{\sigma_C>n\}] \to_{n\to\infty} 0\end{aligned}$$

4. Thus, $\pi(g)=\pi^0_C(g)$, that is $\Delta\pi(g)=0$. This implies $\pi=\pi^0_C$ provided that $g(x)>0$ for $\Delta\pi$ almost all $x\in\mathsf{X}$. But this follows from the assumption since π dominates $\Delta\pi$. Finally, $\pi=\pi^0_C$ and the last equality of the proposition follows from $\pi^1_C=\pi^0_C P=\pi P=\pi$.

2.6 (i) For $p\in\mathbb{N}$, $\sigma_A^{(p+1)}=\sigma_A^{(p)}+\sigma_A\circ\theta_{\sigma_A^{(p)}}$ on $\{\sigma_A^{(p)}<\infty\}$. Applying the strong Markov property yields

$$\begin{aligned}\mathbb{P}_x(\sigma_A^{(p+1)} < \infty) &= \mathbb{P}_x\left(\sigma_A^{(p)} < \infty, \sigma_A \circ \theta_{\sigma_A^{(p)}} < \infty\right) \\ &= \mathbb{E}_x\left[\mathbb{1}\left\{\sigma_A^{(p)} < \infty\right\}\mathbb{P}_{X_{\sigma_A^{(p)}}}(\sigma_A < \infty)\right] \le \delta \mathbb{P}_x(\sigma_A^{(p)} < \infty)\ .\end{aligned}$$

By induction, we obtain $\mathbb{P}_x(\sigma_A^{(p)} < \infty) \le \delta^p$ for every $p \in \mathbb{N}^*$ and $x \in A$. Thus, for $x \in A$,

$$U(x,A) = \mathbb{E}_x[N_A] \le 1 + \sum_{p=1}^{\infty} \mathbb{P}_x(\sigma_A^{(p)} < \infty) \le (1-\delta)^{-1}\ .$$

Since by **??** for all $x \in \mathsf{X}$, $U(x,A) \le \sup_{y \in A} U(y,A)$, (2.1) follows.

(ii) By **??**, $\mathbb{P}_x(\sigma_A^{(n)} < \infty) = 1$ for every $n \in \mathbb{N}$ and $x \in A$. Then,

$$\mathbb{P}_x(N_A = \infty) = \mathbb{P}_x\left(\bigcap_{n=1}^{\infty}\{\sigma_A^{(n)} < \infty\}\right) = 1\ .$$

2.7 1. Using the Markov property,

$$\begin{aligned}P^n f(x) = \mathbb{E}_x[f(X_n)] &= \mathbb{E}_x[\mathbb{1}\{n \le \sigma_A\} f(X_n)] + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{\sigma_A = j\} f(X_n)] \\ &= {}_A^nPf(x) + \sum_{j=1}^{n-1} \mathbb{E}_x\left[\mathbb{1}\{\sigma_A = j\}\mathbb{E}_{X_j}[f(X_{n-j})]\right] \\ &= {}_A^nPf(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{\sigma_A \ge j\}\mathbb{1}_A(X_j) P^{n-j} f(X_j)] \\ &= {}_A^nPf(x) + \sum_{j=1}^{n-1} {}_A^jP(\mathbb{1}_A \times P^{n-j} f)(x)\ . \end{aligned} \tag{2.5}$$

2. The last exit decomposition is established analogously.

$$\begin{aligned}P^n f(x) &= \mathbb{E}_x[f(X_n)] \\ &= \mathbb{E}_x[\mathbb{1}_{\{n \le \sigma_A\}} f(X_n)] + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}\{X_j \in A, X_{j+1} \notin A, \dots, X_{n-1} \notin A\} f(X_n)] \\ &= {}_A^nPf(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}_A(X_j)\mathbb{E}_{X_j}[\mathbb{1}\{X_1 \notin A, \dots, X_{n-j-1} \notin A\} f(X_{n-j})]] \\ &= {}_A^nPf(x) + \sum_{j=1}^{n-1} \mathbb{E}_x[\mathbb{1}_A(X_j)\, {}_A^{n-j}Pf(X_j)] \\ &= {}_A^nPf(x) + \sum_{j=1}^{n-1} P^j(\mathbb{1}_A \times {}_A^{n-j}Pf)(x)\ . \end{aligned} \tag{2.6}$$