Chapter 4
Exercices Week 3

4.1 Dirichlet and Poisson problems

Definition 4.1 (Dirichlet Problem) *Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$, $A \in \mathscr{X}$ and $g \in \mathbb{F}_+(\mathsf{X})$. A nonnegative function $u \in \mathbb{F}_+(\mathsf{X})$ is a solution to the Dirichlet problem if*

$$u(x) = \begin{cases} g(x) , & x \in A , \\ Pu(x) , & x \in A^c . \end{cases} \tag{4.1}$$

For $A \in \mathscr{X}$, we define a submarkovian kernel P_A for $x \in \mathsf{X}$ and $B \in \mathscr{X}$ by

$$P_A(x,B) = \mathbb{E}_x[\mathbb{1}_{\{\tau_A<\infty\}} \mathbb{1}_B(X_{\tau_A})] = \mathbb{P}_x(\tau_A < \infty, X_{\tau_A} \in B) , \tag{4.2}$$

which is the probability that the chain starting from x eventually hits the set $A \cap B$.

4.1. For any $A \in \mathscr{X}$ and $g \in \mathbb{F}_+(\mathsf{X})$, the function $P_A g$ is a solution to the Dirichlet problem (4.1)

Definition 4.2 (Poisson problem) *Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$, $A \in \mathscr{X}$ and $f : A^c \to \mathbb{R}_+$ be a measurable function. A nonnegative function $u \in \mathbb{F}_+(\mathsf{X})$ is a solution to the Poisson problem if*

$$u(x) = \begin{cases} 0 , & x \in A , \\ Pu(x) + f(x) , & x \in A^c . \end{cases} \tag{4.3}$$

For $A \in \mathscr{X}$ and $h \in \mathbb{F}_+(\mathsf{X})$ define

$$G_A h(x) = \mathbb{1}_{A^c}(x) \mathbb{E}_x \left[\sum_{k=0}^{\tau_A - 1} h(X_k) \right] = \mathbb{E}_x \left[\sum_{k=0}^{\tau_A - 1} h(X_k) \right] , \tag{4.4}$$

where we have used the convention $\sum_{k=0}^{-1} \cdot = 0$. Note that $G_A h$ is nonnegative but we do not assume that it is finite.

4.2. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$, $A \in \mathscr{X}$ and $f : A^c \to \mathbb{R}_+$ be a measurable function. The function $G_A f$ is a solution to the Poisson problem (4.3).

4.3. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$ and $A \in \mathscr{X}$. Let $f \in \mathbb{F}_+(A, \mathscr{X}_A)$ and $g \in \mathbb{F}_+(A^c, \mathscr{X}_{A^c})$.

1. Show that the function $P_A g + G_A f$ is a solution to the Poisson-Dirichlet problem

$$u(x) = \begin{cases} g(x) , & x \in A, \\ Pu(x) + f(x) , & x \in A^c . \end{cases} \tag{4.5}$$

2. Show that if $v \in \mathbb{F}_+(\mathsf{X})$ satisfies

$$v(x) \geq \begin{cases} g(x) , & x \in A, \\ Pv(x) + f(x) , & x \in A^c, \end{cases} \tag{4.6}$$

then $v \geq P_A g + G_A f$.

4.4. Show that the function $x \mapsto \mathbb{P}_x(\tau_A < \infty)$ is the smallest positive solution to the system

$$v(x) \geq \begin{cases} 1 & \text{if } x \in A , \\ Pv(x) & \text{if } x \notin A . \end{cases}$$

4.5. Show that the function $x \mapsto \mathbb{E}_x[\tau_A]$ is the smallest positive solution to the system

$$v(x) \geq \begin{cases} 0 & \text{if } x \in A , \\ Pv(x) + 1 & \text{if } x \notin A . \end{cases}$$

4.6. Let P be a Markov kernel on $\mathsf{X} \times \mathscr{X}$. Assume that P admits an atom α and an invariant probability measure π.

1. If $\pi(\alpha) > 0$, show that α is recurrent.
2. If α is accessible, show that $\pi(\alpha) > 0$ and α (and hence P) are recurrent.

Solutions to exercises

4.1 If $x \in A$, then by definition, $P_A g(x) = g(x)$. For $x \in \mathsf{X}$, the identity $\sigma_A = 1 + \tau_A \circ \theta_1$ and the Markov property yield

$$\begin{aligned} PP_A g(x) &= \mathbb{E}_x[P_A g(X_1)] = \mathbb{E}_x[\{\mathbb{1}_{\{\tau_A<\infty\}} g(X_{\tau_A})\} \circ \theta_1] \\ &= \mathbb{E}_x[\mathbb{1}_{\{\tau_A \circ \theta_1<\infty\}} g(X_{1+\tau_A \circ \theta_1})] = \mathbb{E}_x[\mathbb{1}_{\{\sigma_A<\infty\}} g(X_{\sigma_A})] \;. \end{aligned}$$

For $x \notin A$, then $\sigma_A = \tau_A$ $\mathbb{P}_x -$a.s. and we obtain

$$PP_A g(x) = \mathbb{E}_x[\mathbb{1}_{\{\tau_A<\infty\}} g(X_{\tau_A})] = P_A g(x) \;.$$

4.2 Set $u(x) = G_A f(x) = \mathbb{E}_x[S]$ where $S = \mathbb{1}_{A^c}(X_0) \sum_{k=0}^{\tau_A-1} f(X_k)$. By convention $u(x) = 0$ for $x \in A$. Applying the Markov property and the relation $\sigma_A = 1 + \tau_A \circ \theta_1$, we obtain

$$\begin{aligned} Pu(x) &= \mathbb{E}_x[u(X_1)] = \mathbb{E}_x[\mathbb{E}_{X_1}[S]] = \mathbb{E}_x[\mathbb{E}_x[S \circ \theta_1 \mid \mathscr{F}_1]] \\ &= \mathbb{E}_x[S \circ \theta_1] = \mathbb{E}_x\left[\mathbb{1}_{A^c}(X_1) \sum_{k=1}^{\tau_A \circ \theta_1} f(X_k)\right] = \mathbb{E}_x\left[\sum_{k=1}^{\sigma_A-1} f(X_k)\right] \;, \end{aligned} \tag{4.7}$$

where the last equality follows from $\mathbb{1}_A(X_1) \sum_{k=1}^{\sigma_A-1} f(X_k) = 0$. For $x \notin A$, $\sigma_A = \tau_A$ $\mathbb{P}_x -$a.s. and thus

$$f(x) + Pu(x) = f(x) + \mathbb{E}_x\left[\sum_{k=1}^{\sigma_A-1} f(X_k)\right] = \mathbb{E}_x\left[\mathbb{1}_{A^c}(X_0) \sum_{k=0}^{\tau_A-1} f(X_k)\right] = u(x) \;.$$

4.3
1. (4.5) follows by combining Exercise 4.1 with Exercise 4.2.
2. Assume now that (4.6) holds. Eq. (4.6) implies

$$Pv + f\mathbb{1}_{A^c} + g\mathbb{1}_A \le v + \mathbb{1}_A Pv \;.$$

Applying Theorem 4.3.1 (in the book) with $V_n = v$, $Z_n = f\mathbb{1}_{A^c} + g\mathbb{1}_A$, $g = \mathbb{1}_A Pv$ and $\tau = \tau_A$, we obtain for all $x \in A^c$,

$$\begin{aligned} P_A g(x) + G_A f(x) &= \mathbb{E}_x\left[\mathbb{1}_{\{\tau_A<\infty\}} g(X_{\tau_A})\right] + \mathbb{E}_x\left[\sum_{k=0}^{\tau_A-1} f(X_k)\right] \\ &\le \mathbb{E}_x\left[\mathbb{1}_{\{\tau_A<\infty\}} v(X_{\tau_A})\right] \\ &\quad + \mathbb{E}_x\left[\sum_{k=0}^{\tau_A-1} \{f(X_k)\mathbb{1}_{A^c}(X_k) + \mathbb{1}_A(X_k) g(X_k)\}\right] \\ &\le v(x) + \mathbb{E}_x\left[\sum_{k=0}^{\tau_A-1} \mathbb{1}_A(X_k) Pv(X_k)\right] = v(x). \end{aligned}$$

On the other hand, $v(x) \ge g(x) = P_A g(x) + G_A f(x)$ for $x \in A$ by construction.

4.4 Apply Exercise 4.3 with $g = \mathbb{1}_A$ and $f = 0$.

4.5 We apply Exercise 4.3 with $g = 0$ and $f = \mathbb{1}_{A^c}$. In that case, the solution is given by

$$\mathbb{1}_{A^c}(x)\mathbb{E}_x\left[\sum_{k=0}^{\tau_A-1}\mathbb{1}_{A^c}(X_k)\right] = \mathbb{1}_{A^c}(x)\mathbb{E}_x[\tau_A] = \mathbb{E}_x[\tau_A] \; .$$

4.1 (i) Since π is invariant, we have

$$\pi U(\alpha) \; = \sum_{n=0}^{\infty} \pi P^n(\alpha) = \sum_{n=0}^{\infty} \pi(\alpha) \; . \tag{4.8}$$

Therefore, if $\pi(\alpha) > 0$ the atomic version of the maximum principle yields

$$\infty = \pi U(\alpha) = \int_{\mathsf{X}} \pi(\mathrm{d}y) U(y,\alpha) \leq U(\alpha,\alpha) \int_{\mathsf{X}} \pi(\mathrm{d}y) = U(\alpha,\alpha) \; .$$

(ii) Since α is an accessible atom, $K_{a_\varepsilon}(x,\alpha) > 0$ for all $x \in \mathsf{X}$ and $\varepsilon \in (0,1)$. Therefore, we get that

$$\pi(\alpha) = \pi K_{a_\varepsilon}(\alpha) = \int_{\mathsf{X}} \pi(\mathrm{d}x) K_{a_\varepsilon}(x,\alpha) > 0 \; .$$

Therefore α is recurrent by 1 and therefore P is recurrent.