During Lecture 1

$$\underbrace{\delta_x P}(A) = \iint \underbrace{\delta_x(dx_0)}\, P(x_0, dx_1)\, \mathbb{1}_A(x_1).$$

$$= \int \delta_x(dx_0) \underbrace{\int P(x_0, dx_1)\, \mathbb{1}_A(x_1)}_{P(x_0, A).}$$

$$\int \nu(dx)\, \mathbb{1}_A(x) = \nu(A).$$

$$= \int \delta_x(dx_0)\, \underbrace{P(x_0, A)} = P(x, A)$$

Ex 7.1 $X_t = \sigma_t Z_t$, $[\sigma_t^2 = \hat{\alpha}_0 + \hat{\alpha}_1 X_{t-1}^2]$, (we know, Z_i iid, $\mathbb{E}(Z_i) = 0$ $\forall i$. $\mathrm{Var}(Z_i) = 1$

$(Z_t) \perp\!\!\!\perp X_0$.

Set $\mathcal{F}_t = \sigma(X_0, Z_{0:t})$.

X_t = deterministic function of $(Z_t, Z_{t-1}, .., Z_0, X_0)$.

$\Rightarrow$ X_t is $\mathcal{F}_t$-adapted

if $\mathbb{P}(X_{t+1} \in A \mid \mathcal{F}_t) = P(X_t, A)$, $\forall A$ then: (X_t) M.C with M.K (Markov kernel) P.

if: $\mathbb{E}(h(X_{t+1}) \mid \mathcal{F}_t) = \int P(X_t, dx_{t+1})\, h(x_{t+1})$ then ______

$\forall h$ bounded or ≥ 0.

$$\mathbb{E}[h(\underbrace{\sigma_{t+1}}_{\sqrt{\alpha_0 + \alpha_1 X_t^2}} Z_{t+1}) \mid X_0, Z_{0:t}] = \mathbb{E}(h(\sqrt{\alpha_0 + \alpha_1 X_t^2}\, Z_{t+1}) \mid X_0, Z_{0:t}].$$

(because: $\sigma_{t+1}^2 = \alpha_0 + \alpha_1 X_t^2$).

$$= \int_{\mathbb{R}} h(\underbrace{\sqrt{\alpha_0 + \alpha_1 X_t^2}\, z}_{x_{t+1}})\, q(z)\, dz$$

$$x_{t+1} = \sqrt{\alpha_0 + \alpha_1 X_t^2}\, z$$
$$dx_{t+1} = \sqrt{\alpha_0 + \alpha_1 X_t^2}\, dz$$

$$= \int_{\mathbb{R}} h(x_{t+1}) \underbrace{q\left(\frac{x_{t+1}}{\sqrt{\alpha_0 + \alpha_1 X_t^2}}\right) \frac{1}{\sqrt{\alpha_0 + \alpha_1 X_t^2}}\, dx_{t+1}}_{P(X_t,\, dx_{t+1}).}$$

$$\text{where } P(x, dy) = \underbrace{q\left(\frac{y}{\sqrt{\alpha_0 + \alpha_1 x^2}}\right) \frac{1}{\sqrt{\alpha_0 + \alpha_1 x^2}}}_{\text{kernel density}}\, dy$$

7.2 $A = \{\lim n^{-1} \sum_{k=0}^{n-1} h(X_k) = c\}$, $\mathbb{P}_\mu(A) = 1 \Leftrightarrow A$ holds $\mathbb{P}_\mu$-a.s

we have $\mathbb{P}_\mu(A^c) = 0 = \int \mu(dx)\, \underbrace{\mathbb{P}_x(A^c)}_{=0 \text{ for } \mu\text{-almost all } x \in X}$ by (1.2)

Setting $g(x)=P_x(A^c)\geq 0$, we have $\int \mu(dx)\, g(x) = 0$,

for all $\varepsilon>0$, we have $\varepsilon\, \mathbb{1}(g>\varepsilon)\leq g$. Integrating with respect to μ: $\varepsilon\, \mu\{g>\varepsilon\}\leq \underbrace{\int \mu(dx)\, g(x)}_{0}$

therefore: $\mu\{g>\varepsilon\}=0=\int \mu(dx)\underbrace{\mathbb{1}(g(x)>\varepsilon)}_{\xrightarrow[\varepsilon\to 0]{} \mathbb{1}(g(x)>0)}$ $\longrightarrow \int \mu(dx)\,\mathbb{1}\{g(x)>0\}=0=\mu(g>0)$

$\Rightarrow \mu\{g=0\}=1-\mu\{g>0\}=1$

$\Rightarrow g(x)=0$ for μ-almost all x

7.4. $X_t=\mu+\phi X_{t-1}+\sigma Z_t$ where $Z_t \overset{iid}{\sim} \mathcal{N}(0,1)$

$X_0\sim \mathcal{N}(\mu_0,\gamma_0^2)$

(1) $X_1 \overset{\mathcal{L}}{=} X_0$

$X_1=\mu+\phi X_0+\sigma Z_1 \Rightarrow \overbrace{\mathbb{E}(X_1)}^{\mu_0}=\mu+\phi\, \overbrace{\mathbb{E}(X_0)}^{\mu_0}+\sigma\times \overbrace{0}^{\mathbb{E}(Z_1)}$.

$\underbrace{\mathrm{Var}(X_1)}_{\gamma_0^2}=\phi^2\, \underbrace{\mathrm{Var}(X_0)}_{\gamma_0^2}+\sigma^2\, \underbrace{\mathrm{Var}(Z_1)}_{1}$

Then: $\left|\begin{array}{l}\mu_0=\mu+\phi\mu_0 \\ \gamma_0^2=\phi^2\gamma_0^2+\sigma^2\end{array}\right.$ $(*)$

(2) Set $\mu_0=\dfrac{\mu}{1-\phi}$, $\gamma_0^2=\dfrac{\sigma^2}{1-\phi^2}$ | Then: $(*)$ is satisfied, $\begin{cases}\mathbb{E}(X_1)=\mathbb{E}(X_0)\\ \mathrm{Var}(X_1)=\mathrm{Var}(X_0)\end{cases}$.

Then: if $X_0\sim\pi$ where $\pi=\mathcal{N}(\mu_0,\gamma_0^2)$ then $X_1\sim\pi$ i.e.: $\boxed{\pi P=\pi}$

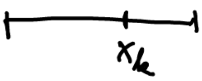
But if X_0 is gaussian then X_1 gaussian.

$X_2\sim\pi P^2$; $X_n\sim \pi P^n$

$X_1\sim\pi P$, $X_0\sim\pi$

$X_1|X_0\sim P(X_0,\cdot)$.

Ex 7.5



Draw $U \nearrow 0$ wp $1/2$, $\searrow 1$ wp $1/2$.

if $U=0$, $X_{k+1}|_{U=0}\sim \mathcal{U}[0,X_k]$.

if $U=1$, $X_{k+1}|_{U=1}\sim \mathcal{U}[X_k,1]$.

$$\mathbb{E}(h(X_{k+1})\mid X_{0:k})=\mathbb{E}(h(X_{k+1})\mid X_k)\overset{?}{=}\int h(x_{k+1})\, P(X_k,dx_{k+1}).$$

$$\mathbb{E}(h(X_{k+1})\mid X_k)=\mathbb{E}\big(h(X_{k+1})\,(\mathbb{1}(U=0)+\mathbb{1}(U=1))\mid X_k\big).$$

$$=\frac{1}{2}\int h(x)\,\frac{\mathbb{1}_{[0,X_k]}(x)}{X_k}\,dx+\frac{1}{2}\int h(x)\,\frac{\mathbb{1}_{[X_k,1]}(x)}{1-X_k}\,dx.$$

$$=\int h(x)\underbrace{\left[\frac{1}{2}\,\frac{\mathbb{1}_{[0,X_k]}(x)}{X_k}+\frac{1}{2}\,\frac{\mathbb{1}_{[X_k,1]}(x)}{1-X_k}\right]dx}_{P(X_k,dx)}.$$

$P(X_k, dx) = k(x_q, x)\,dx$. where $k(x,y) = \frac{1}{2}\frac{\mathbb{1}_{[0,x]}(y)}{x} + \frac{1}{2}\frac{\mathbb{1}_{[x,1]}(y)}{1-x}$

2) $\left[\varepsilon_t \begin{smallmatrix} \nearrow 1 & \text{wp } 1/2 \\ \searrow 0 & \text{wp } 1/2 \end{smallmatrix} \right.$ $\left[U_t \sim \mathcal{U}[0,1] \right.$.

$$X_t = \begin{cases} \underline{X_{t-1}U_t} & \text{if } \varepsilon_t = 1 \\ X_{t-1} + \underbrace{U_t}_{\sim\mathcal{U}[0,1)}\underline{(1-X_{t-1})} & \text{if } \varepsilon_t = 0 \end{cases}$$

| if $U \sim \mathcal{U}[0,1]$.
then $a + bU \sim \mathcal{U}[a, a+b]$.

3)

$$k(x,y) = \frac{1}{2}\frac{1}{x}\mathbb{1}_{]0,x[}(y) + \frac{1}{2}\frac{1}{1-x}\mathbb{1}_{]x,1[}(y).$$

$X_0 \sim p(x)$

$$X_1 \sim \int_0^1 \underbrace{p(x)\,dx}\ \underbrace{k(x,y)} = \int_0^1 dx\ p(x)\left[\frac{1}{2}\frac{1}{x}\overbrace{\mathbb{1}_{]0,x[}(y)}^{\mathbb{1}_{]0<y<x]}} + \frac{1}{2}\frac{1}{1-x}\overbrace{\mathbb{1}_{]x,1[}(y)}^{\mathbb{1}\{x<y<1\}}\right]$$

$$= \frac{1}{2}\int_y^1 \frac{p(x)}{x}\,dx + \frac{1}{2}\int_0^y \frac{p(x)}{1-x}\,dx.$$

if $p(x)\,dx$ is a stationary distribution then: X_0 and X_1 have the same density so: $p(y) = \frac{1}{2}\int_y^1 \frac{p(x)}{x}\,dx + \frac{1}{2}\int_0^y \frac{p(x)}{1-x}\,dx.$

4)

$$p'(y) = \frac{1}{2}\times\left(-\frac{p(y)}{y}\right) + \frac{1}{2}\left(\frac{p(y)}{1-y}\right)$$

$$\frac{p'(y)}{p(y)} = -\frac{1}{2y} + \frac{1}{2(1-y)}.$$

$$\ln p(y) = -\frac{1}{2}\ln y - \frac{1}{2}\ln(1-y) + D. \quad (D \text{ constant}).$$

$$\ln p(y) = \ln\frac{1}{\sqrt{y(1-y)}} + D.) \Rightarrow$$

$$p(y) = \frac{2\,e^D}{2\sqrt{y(1-y)}} \quad e^D = C$$

$$(\operatorname{Arcsin} u)' = \frac{1}{\sqrt{1-u^2}}$$

$$= \frac{2C}{2\sqrt{y(1-y)}}.$$ since $$\underline{(\operatorname{Arcsin}(\sqrt{z}))'} = \frac{1}{\sqrt{1-(\sqrt{z})^2}}\times\frac{1}{2\sqrt{z}} = \frac{1}{2\sqrt{z(1-z)}}.$$

$$\Rightarrow \int_0^z p(y)\,dy = 2C\operatorname{Arcsin}(\sqrt{z}).$$

5) $\int_0^1 p(y)\,dy = 1 = 2c \underbrace{\text{Arcsin}(1)}_{\pi/2}$. $\Rightarrow$ $\boxed{c = 1/\pi}$.

$$p(y) = \frac{1}{\pi\sqrt{y(1-y)}}.$$

Ex 7.6

$$\pi(dx\,dy) = \pi(x,y)\,\lambda_X(dx)\,\lambda_Y(dy).$$

$$\pi(y|x) = \frac{\pi(x,y)}{\pi(x)} \quad \text{where } \pi(x) = \int \pi(x,y)\,\lambda_Y(dy).$$

$$\begin{pmatrix} X_k \\ Y_k \end{pmatrix} \longrightarrow \begin{pmatrix} X_{k+1} \\ Y_{k+1} \end{pmatrix} \qquad \left[\begin{array}{l} X_{k+1} = X_k \\ Y_{k+1} \sim \pi(y'|X_k). \end{array}\right.$$

$$P((x,y), dx'\,dy') = \delta_x(dx')\,\pi(y'|x)\,\lambda_Y(dy').$$

$$\overbrace{\pi(x,y)\,\lambda_X(dx)\lambda_Y(dy)\;P((x,y), dx'dy')}^{} \overset{?}{=} \overbrace{\pi(x',y')\,dx'dy'\;P((x',y'), dx\,dy)}^{\mu(dx\,dy\,dx'dy')}.$$

$$\iiiint h((x,y),(x',y'))\;\pi(x,y)\;\delta_x(dx')\,\pi(y'|x)\,\lambda_Y(dy')\;\lambda_X(dx)\,\lambda_Y(dy)$$

$$= \iiint h((x,y),(x,y'))\;\underbrace{\pi(x,y)}_{\pi(x)\,\pi(y|x).}\;\pi(y'|x)\;\lambda_Y(dy')\,\lambda_X(dx)\,\lambda_Y(dy)$$

$$= \iiint h((x,y),(x,y'))\;\pi(x)\;\pi(y|x)\,\pi(y'|x).\;\lambda_Y(dy')\,\lambda_Y(y)\,\lambda_X(dx) \quad (*)$$

Moreover: $\int\cdots\int h((x,y),(x',y'))\;\mu(dx\,dy\,dx'\,dy').$

$$= \iiint h((x',y),(x',y'))\;\pi(x')\;\pi(y'|x')\,\pi(y|x')\;\lambda_Y(dy')\,\lambda_Y(y)\;\lambda_X(dx). \quad (**)$$

By $(*)$ and $(**)$, we have that P is π-reversible.

$$\overbrace{\mu(dx)\,\delta_x(dx')}^{} = \overbrace{\mu(dx')\,\delta_{x'}(dx)}^{}$$

$$\pi(x,y)\,\lambda_X(dx)\lambda_Y(dy)\;\underline{\delta_x(dx')}\;\pi(y'|x)\,\lambda_Y(dy').$$

$$= \underbrace{\pi(x)\,\delta_x(dx')}_{} \times \pi(y|x)\,\lambda_Y(dy) \times \pi(y'|x)\,\lambda_Y(dy').$$

$$= \pi(x')\,\delta_{x'}(dx). \times \pi(y|x')\,\lambda_Y(dy) \times \pi(y\,|x')\,\lambda_Y(dy').$$

7.7

$$P^{MH}_{\langle \pi, Q\rangle}(x, dy) = Q(x, dy)\,\alpha(x,y) + \bar{\alpha}(x)\,\delta_x(dy).$$

since $\left[\pi(dx)\left[\bar{\alpha}(x)\,\delta_x(dy)\right] = \pi(dy)\,\bar{\alpha}(y)\,\delta_y(dx).\right]$

$$\int h(x,y)\,\pi(dx)\,\bar{\alpha}(x)\,\delta_x(dy) = \int h(x,x)\,\pi(dx)\,\bar{\alpha}(x)$$

$$\int h(x,y)\,\pi(dy)\,\bar{\alpha}(y)\,\delta_y(dx) = \int h(y,y)\,\pi(dy)\,\bar{\alpha}(y).$$

The detailed balance condition is:

(a) $\pi(x)\,q(x,y)\,\alpha(x,y) = \pi(y)\,q(y,x)\,\alpha(y,x).$

(b) $\alpha(x,y) = f\left(\dfrac{\pi(y)\,q(y,x)}{\pi(x)\,q(x,y)}\right)$ Set $u = \dfrac{\pi(y)\,q(y,x)}{\pi(x)\,q(x,y)}.$

(a) $\Rightarrow \underbrace{\alpha(x,y)}_{f(u)} = \underbrace{\dfrac{\pi(y)\,q(y,x)}{\pi(x)\,q(x,y)}}_{u}\,\alpha(y,x).$

$f(u) = u\, f(1/u) \quad \forall u \geq 0.$

(2) if $\forall u \in (0,1)$, $f(u) = u\, f(1/u)$. (α)

Let $v > 1$ $\quad f(v) \overset{?}{=} v\, f(1/v).$

then: $u = \frac{1}{v} \in]0,1[$, then: $f(u) = u\, f\left(\frac{1}{u}\right)$ by (α).

$\Rightarrow f\left(\frac{1}{v}\right) = \frac{1}{v}\, f(v)$ ie: $f(v) = v\, f\left(\frac{1}{v}\right).$

3)

By (2) $\forall u \in]0,1[$, $f(u) = u\, f(1/u) \leq f(1/u).$

Choose arbitrary: $\begin{cases}]1,\infty[\longrightarrow [0,1]. \\ v \mapsto f(v) \in [0,1]. \end{cases}$

then set $f(u) = u\, f(1/u)$, for all $u \in]0,1[$

and then: $f(u) \in [0,1]$. and it satisfies the detailed balance condition.

7.9

$$X_t = \sum_{i=1}^{p} a_i\, X_{t-i} + \sigma \varepsilon_t.$$

$$Y_{t-1} = \begin{pmatrix} X_{t-1} \\ \vdots \\ X_{t-p} \end{pmatrix} \qquad Y_t = \begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-p+1} \end{pmatrix}$$

$$P(y, dy') =$$

$$y = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}, \quad y' = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \qquad P^{I}(y, dy') = (\underset{>0}{\underline{\quad}})\, dy'_1 \dots dy'_p.$$

Thm: 3.7 Let P Markov kernel with a unique invariant prob. measure π. Then: $(X^{\mathbb{N}}, \mathcal{X}^{\otimes \mathbb{N}}, \mathbb{P}_\pi, S)$ is ergodic

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Let $A \in \mathcal{X}^{\otimes \mathbb{N}}$, such that $\mathbb{1}_A = \mathbb{1}_A \circ S$. where $S(\omega_{0:\infty}) = \omega_{1:\infty}$

Let us show that $\mathbb{P}_\pi(A) = 0$ or 1.

[Assume that $\mathbb{P}_\pi(A) > 0$, then we will show that $\mathbb{P}_\pi(A) = 1$.

$$\mathbb{P}_\pi(A) = \int \pi(dx) \underbrace{\mathbb{P}_x(A)}_{h(x)}. \quad ((1.2)\ p\ 12).$$

Assume that we have shown: $\begin{cases} h(X_0) = h(X_1), & \mathbb{P}_\pi\text{-a.s.} \quad (*). \\ h(X_0) = \mathbb{1}_A, & \mathbb{P}_\pi\text{-a.s.} \quad (**). \end{cases}$

Set $$\pi_A(f) = \frac{\int \pi(dx)\, h(x) f(x)}{\int \pi(dx)\, h(x)} = \frac{\int \pi(dx)\, h(x) f(x)}{\mathbb{P}_\pi(A)}.$$

?
$$\begin{aligned}
\int \pi(dx)\, h(x)\, Pf(x) &= \mathbb{E}_\pi[h(X_0)\, Pf(X_0)] \\
&\overset{(1.3)}{=} \mathbb{E}_\pi[h(X_0)\, \mathbb{E}_\pi[f(X_1) \mid X_0]] \\
&= \mathbb{E}_\pi[\mathbb{E}_\pi[h(X_0) f(X_1) \mid X_0]] \\
&= \mathbb{E}_\pi[h(X_0)\, f(X_1)] \\
&= \mathbb{E}_\pi[h(X_1)\, f(X_1)] \quad \} \text{ we use } (*). \\
&= \mathbb{E}_\pi[h(X_0)\, f(X_0)] \\
&= \int \pi(dx)\, h(x) f(x) \dots
\end{aligned}$$

$$\pi_A(Pf) = \frac{\int \pi(dx) h(x) Pf(x)}{\mathbb{P}_\pi(A)} = \frac{\int \pi(dx)\, h(x) f(x)}{\mathbb{P}_\pi(A)} = \pi_A(f).$$

therefore: $\pi_A(Pf) = \pi_A(f)$. But $\underline{P}$ has a unique invariant prob. $\underline{\pi}$

Thus: $\underline{\pi_A = \pi}$.

$\underline{\text{then}}$: $\underline{P_\pi(A)} = \int \pi(dx)\, h(x) = \pi(h) = \pi_A(h) = \dfrac{\int \pi(dx)\, h^2(x)}{P_\pi(A)} = \dfrac{E_\pi[h^2(X_0)]}{P_\pi(A)}$

$$= \frac{E_\pi(\mathbb{1}_A^2)}{P_\pi(A)} = \frac{E_\pi(\mathbb{1}_A)}{P_\pi(A)} = \frac{P_\pi(A)}{P_\pi(A)} = \underline{1}.$$

* Let us show (*) and (**)

$$(X_0, \dots, X_p, \dots) \overset{P_\pi}{\cong} (X_n, X_{n+1}, \dots)$$

. $h(x) = E_x(\mathbb{1}_A)$.

$$C = E_\pi(|h(X_0) - \mathbb{1}_A|) = E_\pi(|h(X_0) - \mathbb{1}_A(X_{0:\infty})|)$$

$$= E_\pi(|h(X_n) - \mathbb{1}_A(X_{n:\infty})|).$$

$$= E_\pi(|h(X_n) - \underbrace{\mathbb{1}_A \circ S^n}_{\mathbb{1}_A}|).$$

$$= E_\pi(|\underbrace{E_{X_n}(\mathbb{1}_A)} - \mathbb{1}_A|) = 0$$

$\omega \in X^{\mathbb{N}}$. $\omega = (\omega_0, \omega_1, \dots)$

$X_{0:\infty}(\omega) = \omega$

$= E_\pi(\mathbb{1}_A \circ S^n | X_{0:n})$ (Markov property).

$= E_\pi(\mathbb{1}_A | X_{0:n})$. (because A invariant set).

$$= E_\pi[|E_\pi(\mathbb{1}_A | X_{0:n}) - \mathbb{1}_A|] \xrightarrow[n\to\infty]{} 0.$$

if $B \in \sigma(X_{0:n})$, $E_\pi(\mathbb{1}_B | X_{0:n}) = \mathbb{1}_B$.

then: $E_\pi[|E_\pi(\mathbb{1}_A - \mathbb{1}_B | X_{0:n}) - (\mathbb{1}_A - \mathbb{1}_B)|]$

$$\leq E_\pi[E_\pi(|\mathbb{1}_A - \mathbb{1}_B| \,|\, X_{0:n}) + |\mathbb{1}_A - \mathbb{1}_B|]$$

$$\leq 2 E_\pi(|\mathbb{1}_A - \mathbb{1}_B|) \leq \delta$$

$\boxed{A \in \mathcal{X}^{\otimes \mathbb{N}}}$, $B \in \sigma(X_{0:n})$. $\underline{\text{Lemma 3.6}}$.

$\underline{\text{Ex 7.10}}$.

P Markov kernel, with invariant prob. meas. π. (ie $\pi P = \pi$).

We assume: $PV + f \leq V + K$ where $\begin{cases} V \geq 0 \\ f \geq 0 \end{cases}$ functions

($K \geq 0$: constant.

show that $\boxed{\pi(f) < \infty}$.

Proof:

$$\begin{cases} \cancel{P}V + f \le V + K. \\ P^2 V + Pf \le \cancel{P}V + K. \\ P^{k+1}V + P^k f \le P^k V + K. \end{cases}$$

$$\longrightarrow \boxed{\frac{\sum_{k=0}^{n} P^k f}{n+1} \le \frac{V}{n+1} + K.}$$

We will show: $\forall n$, $P^{n+1}V + \sum_{k=0}^{n} P^k f \le V + (n+1)K.$] $H(n)$.

Case: $n=0$: $PV + f \le V + K$. (by assumption).

if true for $n-1$. $\quad P^n V + \sum_{k=0}^{n-1} P^k f \le V + nK.$

$$\Rightarrow P^{n+1} V + \sum_{k=1}^{n} P^k f \le PV + nK. + f - f.$$

$$\le V + (n+1)K - f.$$

$$\Rightarrow P^{n+1} V + \sum_{k=0}^{n} P^k f \le V + (n+1)K. \quad (\text{i.e.: } H(n-1) \Rightarrow H(n))$$

$\pi P = \pi \Rightarrow \pi P^k = \pi$ for all $n \ge 0$,

$$\pi(f \wedge M) = \pi\left[\frac{\sum_{k=0}^{n} P^k (f \wedge M)}{n+1}\right] \le \pi\left[\underbrace{\left(\frac{1}{n+1}\sum_{k=0}^{n} P^k f\right)}_{\le \frac{V}{n+1} + K.} \wedge M\right]$$

$$\pi\left(\frac{\sum_{k=0}^{n} P^k (f\wedge M)}{n+1}\right) = \frac{1}{n+1}\sum_{k=0}^{n} \underbrace{\pi P^k (f \wedge M)}_{\pi f} = \frac{1}{n+1}(n+1)\,\pi(f \wedge M)$$

$\forall n$,

$$\pi(f \wedge M) \le \pi\left[\left(\frac{V}{n+1} + K\right) \wedge M\right] = \int \pi(dx)\left[\left(\frac{V(x)}{n+1} + K\right) \wedge M\right]$$

$$\xrightarrow[n \to +\infty]{} \int \pi(dx)\left[(0 + K) \wedge M\right] = K \wedge M.$$

(by dominated convergence)

Then: $\underbrace{\pi(f \wedge M)}_{\xrightarrow[M \to +\infty]{} \pi(f)} \le \underbrace{K \wedge M}_{\to K.} \quad \forall M.$

(by monotone convergence).

Finally: $\boxed{\pi(f) \le K.}$

$a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$.

$\boxed{7.5}$ $X_t = \sum_{i=1}^{p} a_i X_{t-i} + \sigma \varepsilon_t$, where $\varepsilon_t \sim \mathcal{N}(0,1)$.

$\cdot\ f_{\mu,\sigma}(x)$: density of $\mathcal{N}(\mu, \sigma^2)$.

$$X_t \mid X_{0:t-1} \sim \mathcal{N}\Big(\sum_{i=1}^{p} a_i X_{t-i},\ \sigma^2\Big).$$

$\underbrace{(X_t, X_{t+1}, \dots, X_{t+p})}_{Y_{t+p}} \Big| \underbrace{X_{t-1}, \dots, X_{t-p}}_{Y_{t-1}}$ has the density : $\overbrace{f_{\sum_1^p a_i x_{t-i}, \sigma^2}(x_t)}^{>0} \cdot \overbrace{f_{\sum_1^p a_i x_{t+1-i}, \sigma^2}(x_{t+1})}^{>0} \cdots \overbrace{f_{\dots}(x_{t+p})}^{>0}$

with respect to $\underbrace{dx_t \dots dx_{t+p}}$.

$Y_{t+p} \mid Y_{t-1}$ has a strictly $\oplus$ density wp the lebesgue measure on $\mathbb{R}^p$.

$\mathbb{P}(Y_{t+p} \in A \mid Y_{t-1}) = P^{p+1}(Y_{t-1}, A) > 0$ for all A such that $\lambda_p(A) > 0$

where λ_p is the lebesgue measure on $\mathbb{R}^p$.

By Prop 2.10, (Y_t) has at most 1 invariant prob. measure.
