Ex 7.2: we have: $\mathbb{P}_\mu(A) = 1$ where $A = \{\lim_{n \to +\infty} n^{-1} \sum_{k=0}^{n-1} h(X_k) = C\}$.

$$\underbrace{\mathbb{P}_\mu(A)}_{=1} = \int \mu(dy) \underbrace{\mathbb{P}_y(A)}_{\leq 1} \quad \text{and} \quad \int_X \mu(dy) = 1.$$

$$\mathbb{P}_\mu(A^c) = 0 = \int \mu(dy) \underbrace{\underbrace{\mathbb{P}_y(A^c)}_{\geq 0}}_{= 0 \ \mu\text{-a.s}}.$$

then: there exist Ω_0 s.t $\mu(\Omega_0) = 1$ and $\forall y \in \Omega_0$, $\mathbb{P}_y(A) = 1$.

Ex 7.4:

$$\begin{cases} X_t = \mu + \phi X_{t-1} + \sigma Z_t. & (Z_t) \text{ iid } \quad Z_t \sim \mathcal{N}(0,1) \\ & (Z_t)_{t \geq 1} \perp X_0. \\ X_0 \sim \mathcal{N}(\mu_0, \gamma_0^2). \end{cases}$$

1) Assume $X_1 \stackrel{\mathcal{L}}{=} X_0$, show that: $\begin{cases} \mu + \phi \mu_0 = \mu_0. \\ \phi^2 \gamma_0^2 + \sigma^2 = \gamma_0^2. \end{cases}$

$X_1 = \mu + \phi X_0 + \sigma Z_1 \Rightarrow \mathbb{E} X_1 = \mu + \phi \mathbb{E} X_0 = \mu + \phi \mu_0 = \mu_0$

$\Rightarrow \operatorname{Var} X_1 = \operatorname{Var}(\phi X_0 + \sigma Z_1) = \phi^2 \operatorname{Var} X_0 + \sigma^2 \operatorname{Var}(Z_1) =$
$= \phi^2 \gamma_0^2 + \sigma^2 = \gamma_0^2$

2) $\mu + \phi \mu_0 = \mu_0 \Rightarrow \mu_0(1-\phi) = \mu \Rightarrow \mu_0 = \frac{\mu}{1-\phi}$

$\gamma_0^2(\phi^2 - 1) + \sigma^2 = 0 \Rightarrow \gamma_0^2 = \frac{\sigma^2}{1-\phi^2}$

Therefore $X_0 \sim N\left(\frac{\mu}{1-\phi}, \frac{\sigma^2}{1-\phi^2}\right)$, then $X_1 \stackrel{\mathcal{L}}{=} X_0$

where $F_t = \sigma(X_0, Z_{0:t})$.

- $\mathbb{P}(X_t \in A \mid F_{t-1})$

$= \mathbb{P}(\mu + \phi X_{t-1} + \sigma Z_t \in A \mid X_0, Z_{0:t-1})$

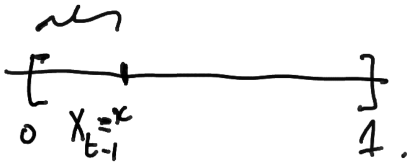
$$= \int \mathbb{1}_A(\underbrace{\mu+\phi X_{t-1}+\sigma z}_{v}) \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz.$$

$v = \mu+\phi X_{t-1}+\sigma z$

$dv = \sigma dz$.

$$= \int \mathbb{1}_A(v) \underbrace{\frac{e^{-\left(\frac{v-\mu-\phi X_{t-1}}{\sigma}\right)^2/2}}{\sqrt{2\pi}} \, \frac{dv}{\sigma}}_{P(X_{t-1},\, dv)}$$

<u>Ex 7.5:</u>



Let ε_t $\begin{cases} 1 & \text{wp } 1/2. \\ 0 & \text{— } 1/2. \end{cases}$

if $\varepsilon'_t = 0$, $X_t \sim \text{Unif}\,[0, X_{t-1}]$.

if $\varepsilon'_t = 1$, $X_t \sim \text{Unif}\,[X_{t-1}, 1]$.

$\mathbb{P}(X_t \in A \mid X_{0:t-1})$.

$= \mathbb{P}(X_t \in A,\ \varepsilon'_t = 0 \mid X_{0:t-1}) + \mathbb{P}(X_t \in A,\ \varepsilon'_t = 1 \mid X_{0:t-1})$.

$$= \frac{1}{2} \int \mathbb{1}_A(y) \frac{\mathbb{1}_{[0,X_{t-1}]}(y)}{X_{t-1}}\, dy + \frac{1}{2} \int \mathbb{1}_A(y) \frac{\mathbb{1}_{[X_{t-1},1]}(y)}{1-X_{t-1}}\, dy.$$

$= \int \mathbb{1}_A(y)\, P(X_{t-1}, dy)$ where:

$$P(x, dy) = \underbrace{\left(\frac{1}{2} \frac{\mathbb{1}_{[0,x]}(y)}{x} + \frac{1}{2} \frac{\mathbb{1}_{[x,1]}(y)}{1-x} \right)}_{k(x,y)} dy.$$

2). $\varepsilon_t = 1 - \varepsilon'_t \sim \text{Be}(1/2)$. Let $U_t \sim \text{Unif}\,[0,1]$ indep. of ε_t.

if $\varepsilon_t = 1$, $X_{t-1} U_t \sim \text{Unif}\,[0, X_{t-1}]$.

if $\varepsilon_t = 0$, $X_{t-1} + U_t(1 - X_{t-1}) \sim \text{Unif}\,[X_{t-1}, 1]$.

3) Recall that μ is stationary for P if : $\mu P = \mu$.

Assume that $\mu(dy) = p(y)\, dy$.

$$\mu(A) = \int \mathbb{1}_A(y)\ \mu(dy) = \int \mathbb{1}_A(y)\, p(y)\, dy.$$

$$\mu P(A) = \iint \mathbb{1}_A(y)\ \underbrace{\mu(dx)}_{p(x)dx}\ \underbrace{P(x,dy)}_{k(x,y)\,dy} = \iint \mathbb{1}_A(y)\, p(x)dx\, \frac{1}{2}\frac{1}{x} \mathbb{1}_{(0,x)}(y) + \frac{1}{2}\frac{1}{1-x}\mathbb{1}_{(x,1)}(y)\ dy.$$

$$= \iint \mathbb{1}_A(y)\, p(y)\, dy$$

$$p(y) = \int p(x)dx\ k(x,y) = \frac{1}{2}\int_{0 \le y \le x} \frac{p(x)}{x}\, dx + \int_{x \le y \le 1} \frac{1}{2}\frac{p(x)}{1-x}\, dx = \frac{1}{2}\int_y^1 \frac{p(x)}{x}\, dx + \frac{1}{2}\int_0^y \frac{p(x)}{1-x}\, dx$$

4) $G(z) = \int_0^z p(y)\, dy.$

$$G'(y) = \frac{1}{2}\int_y^1 \frac{G'(x)}{x}\, dx + \frac{1}{2}\int_0^y \frac{G'(x)}{1-x}\, dx.$$

$$p'(y) = -\frac{1}{2}\frac{p(y)}{y} + \frac{1}{2}\frac{p(y)}{1-y}.$$

$$\Rightarrow \quad \frac{p'(y)}{p(y)} = -\frac{1}{2}\frac{1}{y} + \frac{1}{2}\frac{1}{1-y}. =$$

$$\ln p(y) = C + -\frac{1}{2}\ln y - \frac{1}{2}\ln(1-y).$$

$$= C + \ln\left(\frac{1}{\sqrt{y(1-y)}}\right).$$

$$p(y) = \frac{D}{\sqrt{y(1-y)}}.$$

$$\operatorname{Arcsin}(\sqrt{y})' = \frac{1}{2\sqrt{y}} \times \frac{1}{\sqrt{1-(\sqrt{y})^2}} = \frac{1}{2\sqrt{y(1-y)}}.$$