$\boxed{X_n \overset{w}{\Rightarrow} X}$ $\Longleftrightarrow$ (a) or (b) or (c) or (d).

(a) : $\forall h$ bounded and continuous, $\mathbb{E}(h(X_n)) \underset{n\to\infty}{\longrightarrow} \mathbb{E}(h(X))$.

(b) : $\forall$ set A s.t. $\mathbb{P}(X \in \partial A) = 0$, $\underbrace{\mathbb{E}(\mathbb{1}_A(X_n))}_{\mathbb{P}(X_n \in A)} \underset{n\to+\infty}{\longrightarrow} \mathbb{E}(\mathbb{1}_A(X))$

$\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A)$

(c) $\forall x \in \mathbb{R}$ s.t. $\mathbb{P}(X = x) = 0$, $\underbrace{\mathbb{E}(\mathbb{1}_{]-\infty,x]}(X_n))}_{\mathbb{P}(X_n \leq x)} \longrightarrow \mathbb{E}(\mathbb{1}_{]-\infty,x]}(X))$.

$\mathbb{P}(X_n \leq x) \underset{n\to\infty}{\longrightarrow} \mathbb{P}(X \leq x)$

$\downarrow$ cdf (cumulative distribution function).

(d) $\forall u \in \mathbb{R}$, $\mathbb{E}(\underbrace{e^{iuX_n}}_{h_u(X_n)}) \underset{n\to+\infty}{\longrightarrow} \mathbb{E}(\underbrace{e^{iuX}}_{h_u(X)})$. (Characteristic function).

$\boxed{X_n \xrightarrow{\mathbb{P}\text{-prob}} X}$ $\Longleftrightarrow$ $\forall \varepsilon > 0$, $\mathbb{P}(|X_n - X| \geq \varepsilon) \underset{n\to+\infty}{\longrightarrow} 0$.

$\boxed{X_n \xrightarrow{\mathbb{P}\text{-a.s}} X}$. $\Longleftrightarrow$ $\mathbb{P}\left(\lim_{n\to+\infty} X_n = X\right) = 1$.

Prop:

If: $\begin{cases} X_n \overset{w}{\longrightarrow} X. \\ X_n \xrightarrow{\mathbb{P}\text{-prob}} X. \\ X_n \xrightarrow{\mathbb{P}\text{-a.s}} X \end{cases}$ then: $\forall f$ continuous, $\begin{cases} f(X_n) \overset{w}{\longrightarrow} f(X). \\ f(X_n) \xrightarrow{\mathbb{P}\text{-prob}} f(X) \\ f(X_n) \xrightarrow{\mathbb{P}\text{-a.s}} f(X). \end{cases}$

We have: $\boxed{X_n \xrightarrow{\mathbb{P}\text{-a.s}} X \Rightarrow X_n \xrightarrow{\mathbb{P}\text{-prob}} X \Rightarrow X_n \overset{w.}{\Rightarrow} X.}$

Chap 3.

Strong Law of Large Numbers:

If : $\begin{cases} (1)\ (X_i)_{i\geq 1} & \text{are i.i.d (independent and identically distributed)} \\ (2)\ \mathbb{E}(|X_1|) < \infty. \end{cases}$

then: $\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mathbb{E}(X_1)$. $\mathbb{P}$-a.s.

Central limit theorem.

If : $\begin{cases} (1)\ (X_i)_{i\geq 1.} & \text{are i.i.d.} \\ (2)\ \mathbb{E}(X_1^2) < \infty. \end{cases}$

then:

$$Z_{n.} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_1)}{\sqrt{\frac{Var(X_1)}{n}}} \xrightarrow{\mathcal{L}} Z \quad \text{where } Z \sim \mathcal{N}(0,1).$$

Equivalently : $\tilde{Z}_n = \sqrt{n}\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i}_{\overline{X}_n.} - \mathbb{E}(X_1)\right) \xrightarrow{\mathcal{L}} \tilde{Z}$ where $\tilde{Z} \sim \mathcal{N}(0, Var(X_1))$.

$$\mathbb{E}(Z_{n.}) = \frac{1}{\sqrt{\frac{Var(X_1)}{n}}} \mathbb{E}\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_1)}_{\underbrace{\frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}(X_i)}_{\mathbb{E}(X_1).} - \mathbb{E}(X_1)}_{\underbrace{\mathbb{E}(X_1) - \mathbb{E}(X_1).}_{=0.}}}\right) = 0$$

$$Var(Z_n) = \frac{1}{\frac{Var(X_1)}{n}} Var\left(\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_1).}_{\underbrace{\boxed{Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}}_{\frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)}}\right).$$

because (X_i) are independent

$$= \frac{1}{n^2}\sum_{i=1}^{n} \overbrace{Var(X_i)}^{Var(X_1).} = \frac{1}{n^2}\, n. Var(X_1)$$

$$= \frac{1}{n}\,Var(X_1).$$

$$= \frac{1}{\frac{\cancel{Var(X_1)}}{\cancel{n}}} \cdot \frac{\cancel{Var(X_1)}}{\cancel{n}} = 1.$$

<u>Remark:</u> $Z_n \xrightarrow{\mathcal{L}} Z \Rightarrow \tilde{Z}_n \xrightarrow{\mathcal{L}} \tilde{Z}.$

<u>Indeed:</u> $\left\{\begin{array}{l} Z_n \xrightarrow{\mathcal{L}} Z. \\ f(z) = z.\sqrt{Var(X_1)}. \\ \quad \text{continuous.} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \tilde{Z}_n = f(Z_n). \\ \quad = Z_n \times \sqrt{Var(X_1)} \end{array}\right. \xrightarrow{\mathcal{L}} \underbrace{f(Z)}_{\tilde{Z}.} = Z\sqrt{Var(X_1)}$

Since: $Z \sim \mathcal{N}(0,1)$, $\tilde{Z} = Z\sqrt{Var(X_1)}. \sim \mathcal{N}(\underset{\mathbb{E}(\tilde{Z})}{\underset{\uparrow}{0}}, \underbrace{Var(Z)}_{1.}.\,Var(X_1).)$

$$\tilde{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^N (X_i - \mathbb{E}(X))^2. \quad , \underbrace{\hat{\sigma}_N^2}_{} = \frac{1}{N-1}\sum_{i=1}^N (X_i - \bar{X}_N)^2.$$

Let us show: $\mathbb{E}[\tilde{\sigma}_N^2] = \mathbb{E}(\hat{\sigma}_N^2) = \underbrace{\sigma^2}_{}$ (where: $\underbrace{\sigma^2}_{} = \underbrace{Var(X_1)}_{}$).

$$\tilde{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^N (X_i - \mathbb{E}(X_1))^2. \qquad \text{Set: } \tilde{X}_i = \underbrace{X_i - \mathbb{E}(X_1)}_{}.$$

$$\tilde{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^N \tilde{X}_i^2$$

$$\boxed{\mathbb{E}[\tilde{\sigma}_N^2]} = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \tilde{X}_i^2\right] = \frac{1}{N}\,\mathbb{E}\left[\sum_{i=1}^N \tilde{X}_i^2\right].$$

$$= \frac{1}{N}\sum_{i=1}^N \underbrace{\mathbb{E}[\tilde{X}_i^2]}_{\mathbb{E}[\tilde{X}_1^2]}.$$

$$= \frac{\cancel{N}}{\cancel{N}}\,\mathbb{E}[\tilde{X}_1^2].$$

$$= \mathbb{E}[\tilde{X}_1^2] = \mathbb{E}[(X_1 - \mathbb{E}(X_1))^2] = \underset{= \sigma^2}{Var(X_1)}.$$

$$\boxed{\hat{\sigma}_N^2} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X}_N)^2 \quad . \text{ Setting } \tilde{X}_i = X_i - E(X_1) \qquad \bar{X}_N = \frac{1}{N}\sum_{j=1}^{N} X_j .$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\left(\tilde{X}_i + \cancel{E(X_1)} - \left(\frac{1}{N}\sum_{j=1}^{N}\tilde{X}_j + \cancel{E(X_1)}\right)\right)^2 \qquad \bar{\tilde{X}}_N = \underbrace{\frac{1}{N}\sum_{i=1}^{N}\tilde{X}_i}$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\left(\tilde{X}_i - \bar{\tilde{X}}_N\right)^2$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\left(\tilde{X}_i^2 + \bar{\tilde{X}}_N^2 - 2\tilde{X}_i . \bar{\tilde{X}}_N\right)$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\tilde{X}_i^2 + \frac{N}{N-1}\bar{\tilde{X}}_N^2 - \frac{2}{N-1}\underbrace{\sum_{i=1}^{N}\tilde{X}_i}_{N.\bar{\tilde{X}}_N} . \bar{\tilde{X}}_N$$

$$\hat{\sigma}_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}\tilde{X}_i^2 - \frac{N}{N-1}\bar{\tilde{X}}_N^2$$

$$\boxed{\mathbb{E}(\hat{\sigma}_N^2)} = \mathbb{E}\left[\frac{1}{N-1}\sum_{i=1}^{N}\tilde{X}_i^2 - \frac{N}{N-1}\bar{\tilde{X}}_N^2\right]$$

$$= \frac{1}{N-1}\sum_{i=1}^{N}\underbrace{\mathbb{E}(\tilde{X}_i^2)}_{\underbrace{\mathbb{E}(\tilde{X}_1^2)}_{\underbrace{\mathbb{E}\left[(X_1 - \mathbb{E}(X_1))^2\right]}_{\sigma^2}}} - \frac{N}{N-1}\underbrace{\mathbb{E}(\bar{\tilde{X}}^2)}_{\underbrace{\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N}(X_i - \mathbb{E}(X_1))\right)^2\right]}_{\underbrace{Var\left(\frac{1}{N}\sum_{i=1}^{N}X_i\right)}_{= \frac{Var(X_1)}{N} = \frac{\sigma^2}{N}.}}}$$

$$= \frac{N}{N-1}\sigma^2 - \frac{\cancel{N}}{N-1} \cdot \frac{\sigma^2}{\cancel{N}}.$$

$$= \frac{N-1}{N-1}\sigma^2 = \boxed{\sigma^2.}$$

Recall that $\tilde{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mathbb{E}(X_1))^2$.

By the Law of Large Numbers, $\frac{1}{N}\sum_{i=1}^{N}(X_i - \mathbb{E}(X_1))^2 \xrightarrow[N \to +\infty]{a.s.} \underbrace{\mathbb{E}\left((X_1 - \mathbb{E}(X_1))^2\right)}_{\sigma^2.}$

$$\hat{\sigma}_N^2 = \frac{1}{N-1}\underbrace{\sum_{i=1}^{N}\tilde{X}_i^2}_{,} - \frac{N}{N-1}\underbrace{\tilde{\bar{X}}_N^2}_{\left(\frac{1}{N}\sum_{i=1}^{N} X_i - \mathbb{E}(X_1)\right)^2}$$

$$= \underbrace{\frac{N}{N-1}}_{\xrightarrow[N\to+\infty]{} 1}\left(\underbrace{\frac{1}{N}\sum_{i=1}^{N}(X_i - \mathbb{E}(X_1))^2}_{\xrightarrow{a.s} \sigma^2 \ (LLN)}\right) - \underbrace{\frac{N}{N-1}}_{\xrightarrow[N\to+\infty]{} 1}\left(\underbrace{\frac{1}{N}\sum_{i=1}^{N} X_i - \mathbb{E}(X_1)}_{\xrightarrow[N\to+\infty]{} 0}\right)^2$$

$$\xrightarrow[N\to+\infty]{a.s} 1\times\sigma^2 - 1\times 0 = \sigma^2.$$

An important tool: Slutsky's Lemma:

$$\left[\text{ if } \left| \begin{array}{l} X_n \xrightarrow{\mathcal{L}} X. \\ (Y_n \xrightarrow{\mathbb{P}\text{-prob}} a) \Leftrightarrow (Y_n \xrightarrow{\mathcal{L}} a) \end{array}\right. \quad \text{then } \forall f \text{ continuous, } f(X_n, Y_n) \xrightarrow{\mathcal{L}} f(X, a). \right.$$

Remark: $Y_n \xrightarrow{\mathbb{P}\text{-prob}} Y \Rightarrow Y_n \xrightarrow{\mathcal{L}} Y.$

If Y is a constant called "a" then: $Y_n \xrightarrow{\mathcal{L}} a \Rightarrow Y_n \xrightarrow{\mathbb{P}\text{-prob.}} a.$

Let us show: $\dfrac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{\dfrac{\tilde{\sigma}_N^2}{N}}} \xrightarrow{\mathcal{L}} Z$ where $Z \sim \mathcal{N}(0,1)$

By the Central Limit Theorem, $\boxed{Z_n} = \dfrac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{\dfrac{\sigma^2}{n}}} \boxed{\xrightarrow{\mathcal{L}} Z}$ where $Z \sim \mathcal{N}(0,1)$

Moreover: $\boxed{U_n} = \sqrt{\dfrac{\sigma^2}{\tilde{\sigma}_N^2}} \boxed{\xrightarrow[n\to+\infty]{a.s} 1}$ because: $\tilde{\sigma}_N^2 \to \sigma^2$ ($\tilde{\sigma}_N^2$ strongly convergent).

So: $\boxed{U_n \xrightarrow{\mathbb{P}\text{-prob}} 1} = u.$

Set $f(z,u) = z \times u$, f is continuous

By Slutsky's Lemma,

$$\frac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{\frac{\tilde{\sigma}_N^2}{N}}} = \overbrace{\frac{\bar{X}_n - \mathbb{E}(X_1)}{\sqrt{\frac{\sigma^2}{n}}}}^{Z_n} \times \overbrace{\sqrt{\frac{\sigma^2}{\tilde{\sigma}_N^2}}}^{U_n.} = f(Z_n, U_n) \xRightarrow{\mathcal{L}} \underbrace{f(Z, \underset{=1}{U})}_{Z \times 1 = Z}.$$

. δ-method

If: $\left\{\begin{array}{l} \overbrace{\sqrt{n}(X_n - a)}^{Z_n} \xrightarrow{\mathcal{L}} Z \\ g \text{ differentiable at } a \end{array}\right\} \Rightarrow \underbrace{\sqrt{n}\,(g(X_n) - g(a))}_{\tilde{Z}_n.} \xrightarrow{\mathcal{L}} g'(a) Z.$

$$\tilde{Z}_n = \sqrt{n}\,(g(X_n) - g(a)) = \underbrace{\sqrt{n}\,(X_n - a)}_{Z_n}. \times \underbrace{\frac{g(X_n) - g(a)}{X_n - a.}}_{U_n.} = f(Z_n, U_n)$$

. Now: if $\boxed{U_n \xrightarrow{\mathbb{P}\text{-prob}} g'(a)}$ (*)

then, since: $Z_n \xrightarrow{\mathcal{L}} Z$, we have by Slutsky's Lemma:

$$f(Z_n, U_n) = Z_n \times U_n = \tilde{Z}_n \xrightarrow{\mathcal{L}} f(Z, g'(a)) = g'(a) Z.$$

To conclude, it remains to prove: $U_n \xrightarrow{\mathbb{P}\text{-prob}} g'(a)$.

$$\text{if} \left\{\begin{array}{l} \boxed{X_n \xrightarrow{\mathbb{P}\text{-prob}} a.} \quad (**) \\ x \overset{\varphi}{\mapsto} \left\{\begin{array}{ll} \frac{g(x) - g(a)}{x - a} & x \neq a. \\ g'(a) & x = a. \end{array}\right. \Bigg|\ \varphi \text{ continuous} \end{array}\right] \Rightarrow \begin{array}{l} \varphi(X_n) \xrightarrow{\mathbb{P}\text{-prob}} \varphi(a). \\ \underbrace{\frac{g(X_n) - g(a)}{X_n - a}}_{U_n} \xrightarrow{\mathbb{P}\text{-prob}} g'(a) \end{array}$$

Let us show (**), $\left(\begin{array}{l} Z_n = \sqrt{n}\,(X_n - a) \xrightarrow{\mathcal{L}} Z. \\ U_n = \frac{1}{\sqrt{n}} \xrightarrow{\mathbb{P}\text{-prob}} 0. \end{array}\right.$

$$X_n = Z_n \times \frac{1}{\sqrt{n}} + a = Z_n \times U_n + a := \Psi(Z_n, U_n)$$

where : $\varphi(z,u) = z \times u + a$. is continuous.

By Slutsky's lemma: $\underbrace{\varphi(Z_n, U_n)}_{Z_n U_n + a = X_n} \xrightarrow{\mathcal{L}} \underbrace{\varphi(Z, 0)}_{Z \times 0 + a = a}$.

$$Z_n U_n + a = X_n \xrightarrow{\mathcal{L}} Z \times 0 + a = a.$$

Then $X_n \xrightarrow{\mathcal{L}} a$ which is equivalent to: $\boxed{X_n \xrightarrow{\mathbb{P}\text{-prob}} a.}$

<u>Day 2:</u>

Confidence intervals. $\quad S_N = \frac{1}{N} \sum_{i=1}^{N} f(X_i) \quad , \quad V_N = \frac{1}{N-1} \sum_{i=1}^{N} (f(X_i) - S_N)^2.$

We have seen that: $V_n \xrightarrow{\mathbb{P}\text{-a.s}} \sigma^2 = \mathrm{Var}(f(X_1))$. (Lemma 3.6)

$$A_N = \sqrt{N}\, \frac{S_N - \mathbb{E}(f(X))}{\sqrt{V_N}} = \overbrace{\underbrace{\sqrt{N}\, \frac{(S_N - \mathbb{E}(f(X_1)))}{\sqrt{\sigma^2}}}_{\substack{\xrightarrow{\mathcal{L}} G \\ \text{where } G \sim \mathcal{N}(0,1) \\ (\text{CLT})}}}^{Z_n} \times \overbrace{\underbrace{\sqrt{\frac{\sigma^2}{V_N}}}_{\substack{\xrightarrow{\mathbb{P}\text{-prob}} 1 \\ (\text{by Lemma } 3.6)}}}^{U_n}$$

Slutsky's lemma.

$$\left(\begin{array}{l} Z_n \xrightarrow{\mathcal{L}} G. \\ U_n \xrightarrow{\mathbb{P}\text{-prob}} 1. \end{array} \right) \Rightarrow \underbrace{f(Z_n, U_n)}_{\substack{= Z_n U_n \\ = A_N}} \xrightarrow{\mathcal{L}} \underbrace{f(G,1)}_{= G \times 1 = G}. \quad \text{where } f(z,u) = z \times u. \text{ continuous.}$$

(*).

According to: (*). ,

$$\underbrace{\mathbb{P}\left(-a \le A_N = \sqrt{N} \left| \frac{S_N - \mathbb{E}(f(X))}{\sqrt{V_N}} \right| \le a\right)}_{\mathbb{E}(f(X)) \in I_n.} \xrightarrow[n \to +\infty]{} \underbrace{\mathbb{P}(-a \le G \le a)}_{0.95}.$$

$(a = 1.96)$.

(where $G \sim \mathcal{N}(0,1)$)

$$-a \leq \sqrt{N}\left(\frac{S_N - \mathbb{E}(f(X))}{\sqrt{V_N}}\right) \leq a$$

$$\Leftrightarrow \quad -a\sqrt{V_N} \leq \sqrt{N}\left(S_N - \mathbb{E}(f(X))\right) \leq a\sqrt{V_N}.$$

$$\Leftrightarrow \quad \sqrt{N}\, S_N - a\sqrt{V_N} \leq \sqrt{N}\, \mathbb{E}(f(X)) \leq \sqrt{N}\,(S_N) + a\sqrt{V_N}$$

$$\Leftrightarrow \quad S_N - a\sqrt{\frac{V_N}{N}} \leq \mathbb{E}(f(X)) \leq S_N + a\sqrt{\frac{V_N}{N}}.$$

$$\Leftrightarrow \quad \mathbb{E}(f(X)) \in \underbrace{\left[S_N - a\sqrt{\frac{V_N}{N}},\ S_N + a\sqrt{\frac{V_N}{N}}\right]}_{I_N}.$$

Length of $I_N \sim 2a\sqrt{\frac{\sigma^2}{N}} \xrightarrow[N\to+\infty]{} 0.$

Chapter 4:

. $t \overset{F_Y}{\mapsto} \mathbb{P}(Y \leq t)$ is the cdf (cumulative distribution function) of a random variable Y.

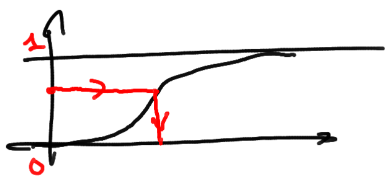
. F_Y is in $[0,1]$. , $\begin{cases} \lim\limits_{t\to-\infty} F_Y(t) = 0. \\ \lim\limits_{t\to+\infty} F_Y(t) = 1. \end{cases}$

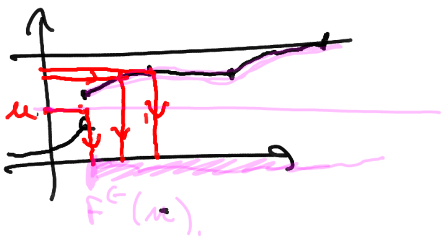
$F_Y \nearrow$. (not always strictly increasing).

. if $a<b$ and $\underbrace{F_Y(a)}_{\mathbb{P}(Y\leq a)} = F_Y(b)$ $\Leftrightarrow$ $\mathbb{P}(a<Y\leq b) = 0$

$\mathbb{P}(Y\leq a) = \mathbb{P}(Y\leq b) = \mathbb{P}(Y\leq a) + \mathbb{P}(a<Y\leq b)$

. if $\lim\limits_{\substack{t\to y\\ t<y}} \underbrace{F_Y(t)}_{\Downarrow\ \mathbb{P}(Y\leq t)} < \underbrace{F_Y(y)}_{\mathbb{P}(Y\leq y)}$. then: $\mathbb{P}(Y=y) = \underbrace{\mathbb{P}(Y\leq y)}_{F_Y(y)} - \lim\limits_{\substack{t\to y\\ t<y}} \underbrace{\mathbb{P}(Y\leq t)}_{F_Y(t)}$





Mathematically speaking,

$F^{\leftarrow}$ $\quad F_Y^{-1}(u) = \inf\{y \,;\, F_Y(y) \geq u\}$

such that.

Proposition 4.2: if $U \sim \mathrm{Unif}\,[0,1]$,

$$F_Y^{-1}(U) \overset{\mathcal{L}}{=} Y.$$

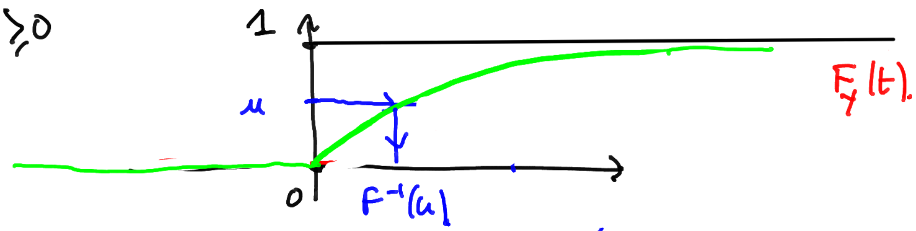
Example: $Y \sim \exp(\lambda)$ ie: Y has the density $f(y) = \lambda e^{-\lambda y}\, \mathbb{1}(y \geq 0)$.

$$\forall t \in \mathbb{R}, \quad \underbrace{\mathbb{P}(Y \leq t)}_{F_Y(t)} = \int_{-\infty}^{t} f(y)\, dy\,.$$

$$= \begin{cases} 0 & \text{if } t \leq 0 \\ \int_0^t f(y)\,dy & \text{if } t \geq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t \leq 0. \\ \int_0^t \lambda e^{-\lambda y}\,dy = \left[-e^{-\lambda y}\right]_0^t = -e^{-\lambda t} + 1 & \text{if } t \geq 0 \end{cases}$$

$$F_Y(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 - e^{-\lambda t} & \text{if } t \geq 0 \end{cases}$$



$f_\lambda(t) = \lambda e^{-\lambda t}$.

$f_\lambda'(t) = -\lambda^2 e^{-\lambda t}$.

$$F_Y(t) = u = 1 - e^{-\lambda t} \quad \underline{ie:} \quad 1 - u = e^{-\lambda t}$$

$\underline{ie:}$ $\ln(1-u) = -\lambda t$.

$\underline{ie:}$ $t = -\dfrac{\ln(1-u)}{\lambda}$.

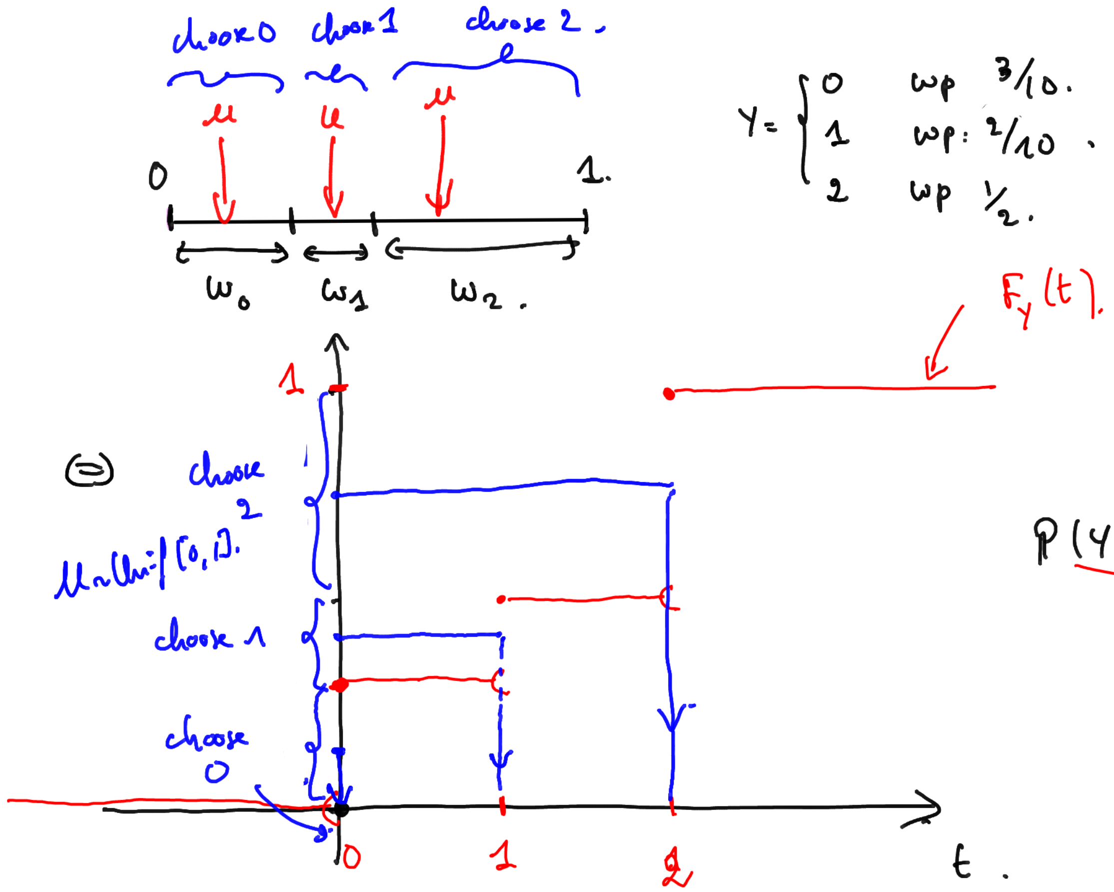
if $U \sim \mathrm{Unif}\,[0,1]$.

then: $-\dfrac{\ln(1-U)}{\lambda} \overset{\mathcal{L}}{=} Y$

(where $Y \sim \exp(\lambda)$)

if $V \sim \mathrm{Unif}\,[0,1]$.

$$-\frac{\ln V}{\lambda} \sim \exp(\lambda).$$



$$Y = \begin{cases} 0 & \text{wp } 3/10 \\ 1 & \text{wp: } 2/10 \\ 2 & \text{wp } 1/2 \end{cases}$$

$P(Y \leq t)$.

Rejection sampling.

f is the target distribution.

Assume that
- $\forall x \in \mathbb{R}$, $f(x) \leq M g(x)$, ie. $\frac{f(x)}{Mg(x)} \leq 1$.
- You can sample from g.

Algorithm:

- Sample $X \sim g$. and $U \sim \mathrm{Unif}[0,1]$.
- While $U > \frac{f(X)}{Mg(X)}$, do: sample $X \sim g$, $U \sim \mathrm{Unif}[0,1]$.
- output $Y = X$.

ie: (X_i, U_i) iid, X_i indep. of U_i, $X_i \sim g$, $U_i \sim \mathrm{Unif}[0,1]$.

$$T = \inf \left\{ t \in \mathbb{N}_* \text{ s.t: } U_i \leq \frac{f(X_i)}{Mg(X_i)} \right\}.$$

then: $Y = X_T$.

We have that $Y \sim f$.

Proof: $\forall A$, $\forall k \in \mathbb{N}_* = \{1,2,3,\dots\}$.

$$\mathbb{P}(Y \in A,\ T = k).$$

$$= \mathbb{P}(X_T \in A,\ T = k).$$

$$= \mathbb{P}(X_k \in A,\ T = k) = \mathbb{P}\left(X_k \in A,\ U_k \leq \frac{f(X_k)}{Mg(X_k)},\ U_{k-1} > \frac{f(X_{k-1})}{Mg(X_{k-1})},\ \dots,\ U_1 > \frac{f(X_1)}{Mg(X_1)}\right).$$

$$= \underbrace{\mathbb{P}\left(X_k \in A,\ U_k \leq \frac{f(X_k)}{Mg(X_k)}\right)}_{\iint \mathbb{1}_A(x)\, \mathbb{1}\left(u \leq \frac{f(x)}{Mg(x)}\right) \cdot \underbrace{g(x)\, \mathbb{1}_{[0,1]}(u)}_{\text{joint density of } (X,U)} dx\, du} \times \underbrace{\mathbb{P}\left(U_{k-1} > \frac{f(X_{k-1})}{Mg(X_{k-1})}\right) \times \dots \times \mathbb{P}\left(U_1 > \frac{f(X_1)}{Mg(X_1)}\right)}_{\left(\iint \mathbb{1}\left(u > \frac{f(x)}{Mg(x)}\right) g(x)\, \mathbb{1}_{[0,1]}(u)\, dx\, du\right)^{k-1}}$$

$$= \int_{-\infty}^{+\infty} \mathbb{1}_A(x) \underbrace{\left(\int_0^{\frac{f(x)}{Mg(x)}} du\right)}_{\frac{f(x)}{M\cancel{g(x)}}} \cancel{g(x)}\, dx \times \left[\int_{-\infty}^{+\infty} \underbrace{\left[\int_{\frac{f(x)}{Mg(x)}}^{1} du\right]}_{\left(1 - \frac{f(x)}{Mg(x)}\right)} g(x)\, dx\right]^{k-1}$$

$$= \int_{-\infty}^{+\infty} \mathbb{1}_A(x) \frac{f(x)}{M} dx \times \underbrace{\left[\int_{-\infty}^{+\infty} \left(g(x) - \frac{f(x)}{M}\right) dx\right]^{k-1}}_{\left(1 - \frac{1}{M}\right)^{k-1}} \quad \left(\text{since: } \int_{-\infty}^{+\infty} g(x)\, dx = 1,\ \int_{-\infty}^{+\infty} f(x)\, dx = 1\right)$$

$$= \underbrace{\int_A f(x)\, dx}_{\mathbb{P}(Y \in A)} \times \underbrace{\frac{1}{M} \times \left(1 - \frac{1}{M}\right)^{k-1}}_{\mathbb{P}(T = k)} = \mathbb{P}(Y \in A,\ T = k).$$

ie: $Y \perp\!\!\!\perp T$ (Y is indep. of T).

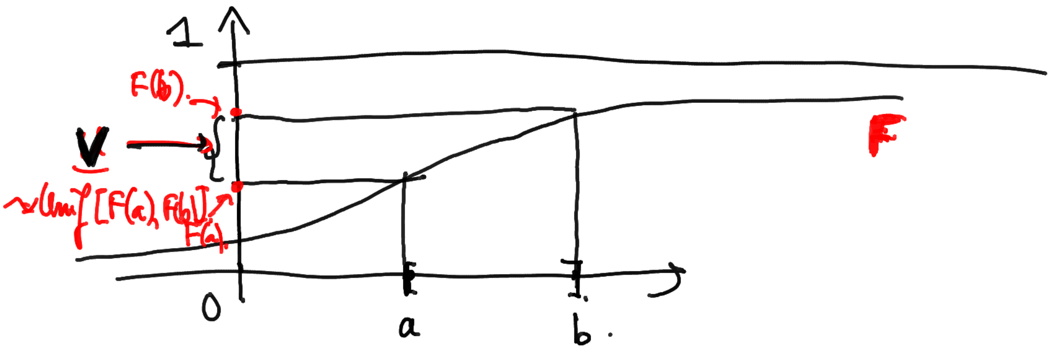
$Y \sim f$, $\quad T \sim \text{Geom}\left(\frac{1}{M}\right)$.

Sampling a conditional distribution. <u>Aim</u>: sample $X|_{X\in[a,b]}$, where $X\sim g$.

1) <u>Method 1</u>.

- Repeat $X\sim g$, until $X\in[a,b]$. Set $Y=X$.

2) <u>Method 2</u>: Sampling by the quantile function.



if $U\sim \mathrm{Unif}[0,1]$.

$$V = F(a) + U(F(b)-F(a)) \sim \mathrm{Unif}([F(a), F(b)]).$$

$$F^{-1}(V) = F^{-1}(F(a)+U(F(b)-F(a))) \sim \frac{g(x)\,\mathbb{1}_{[a,b]}(x)}{\int_a^b g(u)\,du}.$$

<u>Importance sampling methods</u>:

Approximate: $\overbrace{\int f(x)\,h(x)\,dx}^{I}$. where f is a density.

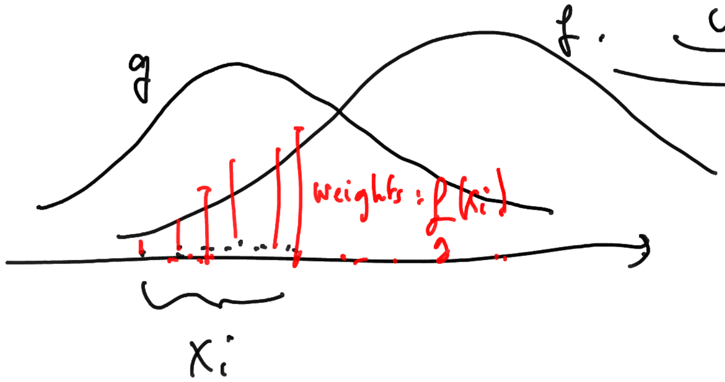
$\underbrace{\int f(x)\,h(x)\,dx}_{\mathbb{E}_f[h(X)]}$ (ie $\mathbb{E}(h(X))$ when $X\sim f$).

$$\underbrace{\int g(x)\,\frac{f(x)}{g(x)}\,h(x)\,dx}_{=\mathbb{E}_g\left[\frac{f(X)}{g(X)}h(X)\right]}.$$

where $X\sim g$.

Sample: X_i iid, $X_1 \sim g$.

Approximate I by: $\displaystyle \frac{1}{N}\sum_{i=1}^{N} \underbrace{\frac{f(X_i)}{g(X_i)}}_{\text{weights}} h(X_i) \xrightarrow[N\to\infty]{LLN} I$



Variant: if $f(x)$ is known up to a multiplicative constant:

$$f(x) = C\,\tilde{f}(x) \quad \text{where } C \text{ is unknown.}$$

$$\frac{\displaystyle\sum_{i=1}^{N} \frac{\tilde{f}(X_i)}{g(X_i)} h(X_i)}{\displaystyle\sum_{i=1}^{N} \frac{\tilde{f}(X_i)}{g(X_i)}} = \frac{\left[\displaystyle\frac{1}{N}\sum_{i=1}^{N} \frac{f(X_i)}{g(X_i)} h(X_i)\right] \to \displaystyle\int \cancel{g(x)}\frac{f(x)h(x)}{\cancel{g(x)}}dx}{\left(\displaystyle\frac{1}{N}\sum_{i=1}^{N} \frac{f(X_i)}{g(X_i)}\right) \to \displaystyle\int \cancel{g(x)}\frac{f(x)}{\cancel{g(x)}}dx.}$$

$$\longrightarrow \frac{\int f(x)h(x)dx}{\underbrace{\int f(x)dx}_{=1}} = \underbrace{\int f(x)h(x)dx}_{I}.$$

- $\displaystyle \frac{1}{N}\sum_{i=1}^{N} \frac{f(X_i)}{g(X_i)} h(X_i) = \hat{I}_N(f)$ approximates $I(f) = \int f(x)h(x)dx$.

$$\mathbb{E}\left[\hat{I}_N(f)\right] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[\frac{f(X_i)}{g(X_i)} h(X_i)\right]. \qquad X_i \sim g.$$

$$= \frac{1}{N}\sum_{i=1}^{N} \int \cancel{g(x)} \frac{f(x)}{\cancel{g(x)}} h(x)dx = \int f(x)h(x)dx = I(f).$$

unbiased estimator.

$$Var\left(\hat{I}_N(f)\right) = Var\left(\frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{g(X_i)}h(X_i)\right).$$

$$= \frac{1}{N^2}\underbrace{Var\left(\sum_{i=1}^{N}\frac{f(X_i)}{g(X_i)}h(X_i)\right)}_{\sum_{i=1}^{N} Var\left(\frac{f(X_i)}{g(X_i)}h(X_i)\right)}.$$

$$\underbrace{\sum_{i=1}^{N} Var\left(\frac{f(X_i)}{g(X_i)}h(X_i)\right)}_{N\,Var\left(\frac{f(X_1)}{g(X_1)}h(X_1)\right)}$$ (because (X_i) indep).

$N\,Var\left(\frac{f(X_1)}{g(X_1)}h(X_1)\right)$ (they have the same Law).

$Y.$

$$= \frac{1}{N}Var\left(\frac{f(X_1)}{g(X_1)}h(X_1)\right).$$

$$\underbrace{\int\left(\frac{f(x)}{g(x)}h(x)\right)^2 g(x)\,dx}_{E(Y^2)} - \underbrace{\left(\int\frac{f(x)}{g(x)}h(x)\cdot g(x)\,dx\right)^2}_{E(Y)^2}.$$

$$= \frac{1}{N}\left[\underbrace{\int\frac{f^2(x)h^2(x)}{g(x)}dx} - \left(\int f(x)h(x)\right)^2\right].$$

Aim: search for density g such that it mimizes: $\displaystyle\int\frac{f^2(x)h^2(x)}{g(x)}dx.$
$\left(\int g(x)dx = 1\right)$

But: $$Var(|Y|) = Var\left(\frac{f(x)}{g(x)}|h(x)|\right) = \int\frac{f^2(x)h^2(x)}{g(x)}dx - \left(\int f(x)|h(x)|dx\right)^2$$
$\geq 0.$

This implies: $$\underbrace{\int f^2(x)\,\frac{h(x)^2}{g(x)}}\,dx \geq \left(\int f(x)\,|h(x)|\,dx\right)^2$$

we attain the lower bound for: $|Y|$ constant

that is: $$\frac{f(x)\,|h(x)|}{g^*(x)} = c \quad \forall x.$$

that is: $$\boxed{g^*(x) = \frac{f(x)\,|h(x)|}{\int f(y)\,|h(y)|\,dy}.}$$ (then: $\int g(x)dx = 1$).

Computersession 2:

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,dt.$$

$$= \left[\frac{1}{\pi}\,\mathrm{Atan}(t)\right]_{-\infty}^{x}.$$

$$= \frac{1}{\pi}\,\mathrm{Atan}(x) - \frac{1}{\pi}\left(-\frac{\pi}{2}\right).$$

$$F(x) = \frac{1}{\pi}\,\mathrm{Atan}(x) + \frac{1}{2} = u$$

$$\frac{1}{\pi}\,\mathrm{atan}(x) + \frac{1}{2} = u \Rightarrow \mathrm{atan}\,x = \left(u - \frac{1}{2}\right)\pi.$$

$$x = \underbrace{\tan\left(\pi u - \frac{\pi}{2}\right)}_{F^{-1}(u)}\left(= -\mathrm{cotan}(\pi u)\right)$$

Draw: $U \sim \mathrm{Unif}\,[0,1]$

Set: $X = \tan\left(\pi U - \frac{\pi}{2}\right).$

For exponential distribution of parameter λ :

$$X \sim \exp(\lambda), \quad \begin{cases} F(x) = \left(1 - e^{-\lambda x}\right) & , \ \forall x \geq 0. \\ F^{-1}(u) = -\dfrac{\ln(1-u)}{\lambda}. \end{cases}$$

$$\mathbb{E}(X) = \frac{1}{\lambda}, \quad \mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

<u>CLT:</u>

$$\underbrace{\sqrt{n}\left(\frac{\overline{X}_n - \frac{1}{\lambda}}{\sqrt{1/\lambda^2}}\right)}_{\sqrt{n}\left(\lambda \overline{X}_n - 1\right)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1).$$

$$\sqrt{n}\left(\lambda \overline{X}_n - 1\right) \xrightarrow{\mathcal{L}} G \text{ where } G \sim \mathcal{N}(0,1).$$

$$\mathbb{P}\big(\underbrace{-a \leq \sqrt{n}\left(\lambda \overline{X}_n - 1\right) \leq a}_{-\frac{a}{\sqrt{n}} \leq \lambda \overline{X}_n - 1 \leq a}\big) \xrightarrow[n \to +\infty]{} \mathbb{P}(-a \leq G \leq \underset{\substack{\| \\ 1.96}}{a}) = 0.95.$$

$$-\frac{a}{\sqrt{n}} \leq \lambda \overline{X}_n - 1 \leq a.$$

$$\Leftrightarrow -\frac{a}{\sqrt{n}\,\overline{X}_n} \leq \lambda - \frac{1}{\overline{X}_n} \leq \frac{a}{X_n}.$$

$$\Leftrightarrow \lambda \in \underbrace{\left[\frac{1}{\overline{X}_n} \pm \frac{a}{\sqrt{n}(\overline{X}_n)}\right]}.$$

$$\Leftrightarrow \lambda \in \underbrace{\left[\frac{1}{\overline{X}_n} - \frac{a}{\sqrt{n}\,\overline{X}_n}, \frac{1}{\overline{X}_n} + \frac{a}{\sqrt{n}\,\overline{X}_n}.\right]}_{a = 1.96.}$$

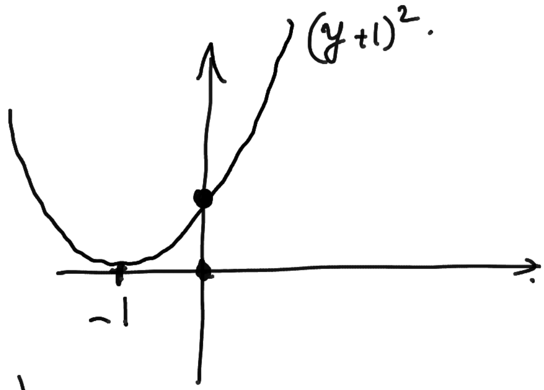
$Y \sim \mathcal{N}(0,1)$. Y has density: $\tilde{f}(y) = \dfrac{e^{-y^2/2}}{\sqrt{2\pi}}$.

$Y \,|_{Y \in \mathbb{R}^+}$ this condition distribution has density: $\dfrac{\tilde{f}(y)\, \mathbb{1}_{\mathbb{R}^+}(y)}{\int \tilde{f}(z)\, \mathbb{1}_{\mathbb{R}^+}(z)\, dz}. = f(y)$

$$f(y) = \frac{\frac{e^{-y^2/2}}{\sqrt{2\pi}} \mathbb{1}_{\mathbb{R}^+}(y)}{\underbrace{\int_{\mathbb{R}^+} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz}_{=\frac{1}{2}}} = 2 \frac{e^{-y^2/2}}{\sqrt{2\pi}} \mathbb{1}_{\mathbb{R}^+}(y).$$

$$f(y) \leq M e^{-y} \qquad \forall y \geq 0.$$

$$\frac{f(y)}{e^{y}} \leq M \qquad \forall y \geq 0.$$



$$M = \sup_{y\geq 0} \frac{f(y)}{e^y} = \sup_{y\geq 0} 2\underbrace{e^{-y^2/2 - y}}_{e^{-\frac{1}{2}(\underbrace{y^2+2y}_{(y+1)^2-1})}}$$

$$= \sup_{y \geq 0} 2 e^{-\frac{1}{2}(y+1)^2 + \frac{1}{2}} = 2.$$

$$= 2e^{-\frac{1}{2}\underbrace{\inf_{y\geq 0}(y+1)^2}_{1} + \frac{1}{2}} = 2e^{-\frac{1}{2}+\frac{1}{2}} = 2e^0 = 2.$$

therefore: $f(y) = 2e^{-y^2/2}\mathbb{1}_{\mathbb{R}^+}(x) \leq \underbrace{2}_{M} \underbrace{e^{-y}}_{g(y)}$.