MCMC Exam

25 October

1 Exercise 1.

Let Q_1, Q_2 be two probability kernels on, respectively, $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and $(\mathbb{R}^-_*, \mathcal{B}(\mathbb{R}^-_*))$. Let π_1, π_2 be two probability measures on, respectively, $\mathcal{B}(\mathbb{R}^+)$ and $\mathcal{B}(\mathbb{R}^-_*)$, such that π_1 is invariant by Q_1 and π_2 invariant by Q_2.

Question 1.1. Let $Q : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ be defined as:

$$\forall x, A \in \mathbb{R} \times \mathcal{B}(\mathbb{R}), \quad Q(x,A) = \mathbb{1}_{\mathbb{R}^+}(x) Q_1(x, A \cap \mathbb{R}^+) + \mathbb{1}_{\mathbb{R}^-_*}(x) Q_2(x, A \cap \mathbb{R}^-_*) .$$

Show that Q is a probability kernel.

Define $\tilde{\pi}_1, \tilde{\pi}_2$ two probability measures on $\mathcal{B}(\mathbb{R})$ as:

$$\forall A \in \mathcal{B}(\mathbb{R}) \quad \tilde{\pi}_1(A) = \pi_1(A \cap \mathbb{R}^+) \quad \text{and} \quad \tilde{\pi}_2(A) = \pi_2(A \cap \mathbb{R}^-_*) .$$

Furthermore, define π_3 a probability measure on $\mathcal{B}(\mathbb{R})$ as $\pi_3 = \frac{1}{2}\tilde{\pi}_1 + \frac{1}{2}\tilde{\pi}_2$.

Question 1.2. Show that $\tilde{\pi}_1, \tilde{\pi}_2, \pi_3$ are invariant for the kernel Q.

Question 1.3. Give an example of an other probability measure π, invariant for Q.

Question 1.4. Let (X_k) be a Markov chain on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with a transition kernel Q. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded, measurable function. Do we know to what quantity will converge:

$$\frac{1}{n+1} \sum_{i=0}^{n} h(X_i) .$$

On what additional information it will depend?

We produce (X_k) by Algorithm 1.

Question 1.5. Write down $\tilde{Q}$ the Markov kernel of (X_k).

In the following, assume that π_1 (respectively π_2) is dominated by the Lebesgue measure on $\mathcal{B}(\mathbb{R}^+)$ (respectively on $\mathcal{B}(\mathbb{R}^-_*)$). We will denote its density p_1 (respectively p_2). We also assume that for all $x > 0$, $p_1(x) = p_2(-x)$.

Question 1.6. Show that π_3 is an invariant probability measure for $\tilde{Q}$.

Question 1.7. Let $A \in \mathcal{B}(\mathbb{R}^+)$ show that for all $x \geqslant 0$ and for all $n \in \mathbb{N}$,

$$\tilde{Q}^n(x, A) \geqslant \frac{1}{2^n} Q_1^n(x, A)\,.$$

Establish a similar lower bound on $\tilde{Q}^n(x, A)$ in the case where $x < 0$.

Question 1.8. On what condition on Q_1 the measure π_3 will be the unique invariant measure for $\tilde{Q}$?

Question 1.9. Propose a modification of the algorithm to sample from $\frac{1}{3}\tilde{\pi}_1 + \frac{2}{3}\tilde{\pi}_2$.

Algorithm 1 Input: $x_0 \in \mathbb{R}$

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X_0 = x_0.
for k ⩾ 0 do
    Sample U_k, in an independent manner, with a uniform distribution on [0, 1].
    if U_k ⩽ 1/2 then
        Sample X_{k+1} from Q(|X_k|, ·)
    else
        Sample X_{k+1} from Q(−|X_k|, ·)
    end if
end for
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