MCMC Exam. Answers

25 October

1 Exercise 1.

Let Q_1, Q_2 be two probability kernels on, respectively, $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and $(\mathbb{R}^-_*, \mathcal{B}(\mathbb{R}^-_*))$. Let π_1, π_2 be two probability measures on, respectively, $\mathcal{B}(\mathbb{R}^+)$ and $\mathcal{B}(\mathbb{R}^-_*)$, such that π_1 is invariant by Q_1 and π_2 invariant by Q_2.

Question 1.1. Let $Q : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ be defined as:

$$\forall x, A \in \mathbb{R} \times \mathcal{B}(\mathbb{R}), \quad Q(x, A) = \mathbb{1}_{\mathbb{R}^+}(x) Q_1(x, A \cap \mathbb{R}^+) + \mathbb{1}_{\mathbb{R}^-_*}(x) Q_2(x, A \cap \mathbb{R}^-_*) .$$

Show that Q is a probability kernel.

For every $A \in \mathcal{B}(\mathbb{R})$ the function $x \mapsto Q(x, A)$ is measurable as a composition (product/sum ...) of measurable functions. Similarly, for every $x \in \mathbb{R}$, the countable additivity of $Q(x, \cdot)$ is immediate and $Q(x, \mathbb{R}) = 1$, thus $Q(x, \cdot)$ is indeed a probability measure and Q is a probability kernel. □

Define $\tilde{\pi}_1, \tilde{\pi}_2$ two probability measures on $\mathcal{B}(\mathbb{R})$ as:

$$\forall A \in \mathcal{B}(\mathbb{R}) \quad \tilde{\pi}_1(A) = \pi_1(A \cap \mathbb{R}^+) \quad \text{and} \quad \tilde{\pi}_2(A) = \pi_2(A \cap \mathbb{R}^-_*) .$$

Furthermore, define π_3 a probability measure on $\mathcal{B}(\mathbb{R})$ as $\pi_3 = \frac{1}{2}\tilde{\pi}_1 + \frac{1}{2}\tilde{\pi}_2$.

Question 1.2. Show that $\tilde{\pi}_1, \tilde{\pi}_2, \pi_3$ are invariant for the kernel Q. Let $A \in \mathcal{B}(\mathbb{R})$ and denote $A_+ = A \cap \mathbb{R}_+$ and $A_- \cap \mathbb{R}^*_-$. It holds that

$$\begin{aligned} \tilde{\pi}_1(A) &= \tilde{\pi}_1(A_+) = \pi_1(A_+) \\ &= \pi_1 Q_1(A_+) = \int_{y \in \mathbb{R}_+} \pi_1(\mathrm{d}y) Q_1(y, A_+) = \int_{y \in \mathbb{R}_+} \tilde{\pi}_1(\mathrm{d}y) Q(y, A_+) . \end{aligned}$$

Now, notice that for $y \geqslant 0$, $Q(y, A_+) = Q(y, A)$ and that $\tilde{\pi}(\mathbb{R}^*_-) = 0$. Hence,

$$\tilde{\pi}_1(A) = \int_{y \in \mathbb{R}_+} \tilde{\pi}_1(\mathrm{d}y) Q(y, A_+) = \int_{y \in \mathbb{R}_+} \tilde{\pi}_1(\mathrm{d}y) Q(y, A) = \int_{y \in \mathbb{R}} \tilde{\pi}_1(\mathrm{d}y) Q(y, A) ,$$

which means that $\tilde{\pi}_1$ is indeed invariant for Q. A symmetric reasoning shows that $\tilde{\pi}_2$ is invariant. Finally, π_3 is invariant as a convex combination of invariant measures.

Question 1.3. Give an example of an other probability measure π, invariant for Q.

Any convex combination of $\tilde{\pi}_1$ and $\tilde{\pi}_2$ works. E.g. $\frac{1}{3}\tilde{\pi}_1 + \frac{2}{3}\tilde{\pi}_2$.

Question 1.4. Let (X_k) be a Markov chain on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with a transition kernel Q. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded, measurable function. Do we know to what quantity will converge:

$$\frac{1}{n+1} \sum_{i=0}^{n} h(X_i) \,.$$

On what additional information it will depend?

As we have seen, the markov kernel Q admits more than one invariant probability measure. Typically, the convergence of this sum will depend on the initialization. For instance, if we initialize in $\mathbb{R}_+$ and π_1 is the unique invariant measure of an irreducible kernel Q_1, then we will stay forever in $\mathbb{R}_+$ and this sum will converge to $\pi_1(h)$.

We produce (X_k) by Algorithm 1.

Question 1.5. Write down $\tilde{Q}$ the Markov kernel of (X_k).

The markov kernel is $\tilde{Q}(x, \mathrm{d}y) = \frac{1}{2} Q(|x|, \mathrm{d}y) + \frac{1}{2} Q(-|x|, \mathrm{d}y)$.

In the following, assume that π_1 (respectively π_2) is dominated by the Lebesgue measure on $\mathcal{B}(\mathbb{R}^+)$ (respectively on $\mathcal{B}(\mathbb{R}^-_*)$). We will denote its density p_1 (respectively p_2). We also assume that for all $x > 0$, $p_1(x) = p_2(-x)$.

Question 1.6. Show that π_3 is an invariant probability measure for $\tilde{Q}$.

Let $A \in \mathcal{B}(\mathbb{R})$ and denote $A_+ = A \cap \mathbb{R}_+$ and $A_- = A \cap \mathbb{R}_-$. It holds

$$\pi_3 \tilde{Q}(A) = \int_{y \in \mathbb{R}} \pi_3(\mathrm{d}y) \tilde{Q}(y, A) = \frac{1}{2} \int_{y \in \mathbb{R}} \pi_3(\mathrm{d}y) Q(|y|, A) + \frac{1}{2} \int_{y \in \mathbb{R}} \pi_3(\mathrm{d}y) Q(-|y|, A) \,.$$

Now,

$$\begin{aligned}
\int_{y \in \mathbb{R}} \pi_3(\mathrm{d}y) Q(|y|, A) &= \frac{1}{2} \int_{y \in \mathbb{R}_+} \tilde{\pi}_1(\mathrm{d}y) Q(y, A) + \frac{1}{2} \int_{y \in \mathbb{R}_-} \tilde{\pi}_2(\mathrm{d}y) Q(-y, A) \\
&= \frac{1}{2} \left(\int_{y \in \mathbb{R}_+} p(y) Q(y, A) + \int_{y \in \mathbb{R}_-} p(-y) Q(-y, A) \right) \\
&= \int_{y \in \mathbb{R}_+} p(y) Q(y, A) \\
&= \int_{y \in \mathbb{R}_+} \tilde{\pi}_1(\mathrm{d}y) Q(y, A) \\
&= \int_{y \in \mathbb{R}} \tilde{\pi}_1(\mathrm{d}y) Q(y, A) = \tilde{\pi}_1(A) \,,
\end{aligned}$$

where for the third equality we have made the change of variables $y \mapsto -y$ in the second integral and in the last equality we have used the invariance of $\tilde{\pi}_1$ for Q.
Similar computations show that $\int_{y \in \mathbb{R}} \pi_3(\mathrm{d}y) Q(-|y|, A) = \tilde{\pi}_2(A)$. Thus, $\pi_3 \tilde{Q}(A) = \frac{1}{2} \tilde{\pi}_1(A) + \frac{1}{2} \tilde{\pi}_2(A) = \pi_3(A)$.

Question 1.7. Let $A \in \mathcal{B}(\mathbb{R}^+)$ show that for all $x \geqslant 0$ and for all $n \in \mathbb{N}$,

$$\tilde{Q}^n(x, A) \geqslant \frac{1}{2^n} Q_1^n(x, A) \,.$$

Establish a similar lower bound on $\tilde{Q}^n(x, A)$ in the case where $x < 0$.

We prove this fact by induction. For, $n = 1$ this equality is immediate by the fact that $\tilde{Q}(x, A) = \frac{1}{2}Q(x, A) + \frac{1}{2}Q(-x, A)$ (x is positive) and the fact that $Q(x, A) = Q_1(x, A)$ ($A \in \mathcal{B}(\mathbb{R}_+)$). Assume that it is true for some $n \in \mathbb{N}$. Then,

$$\tilde{Q}^{n+1}(x, A) = \int_{y\in\mathbb{R}} \tilde{Q}(x, \mathrm{d}y)\tilde{Q}^n(y, A) \geqslant \int_{y\in\mathbb{R}_+} \tilde{Q}(x, \mathrm{d}y)Q^n(y, A) \geqslant \frac{1}{2^n} \int_{y\in\mathbb{R}_+} \tilde{Q}(x, \mathrm{d}y)Q_1^n(y, A)$$

Moreover, since $x > 0$, $\tilde{Q}(x, \mathrm{d}y) = \frac{1}{2}(Q(x, \mathrm{d}y) + Q(-x, \mathrm{d}y)) \geqslant \frac{1}{2}Q(x, \mathrm{d}y)$ and

$$\int_{y\in\mathbb{R}_+} \tilde{Q}(x, \mathrm{d}y)Q_1^n(y, A) \geqslant \frac{1}{2} \int_{y\in\mathbb{R}_+} Q(x, \mathrm{d}y)Q_1^n(y, A)\,.$$

Finally, since the last integral if on $\mathbb{R}_+$ we have $Q(x, \mathrm{d}y) = Q_1(x, \mathrm{d}y)$ which finishes the proof.

If $x < 0$ then, $Q(-x, A) = Q_1(-x, A)$ and $\tilde{Q}(x, A) = \frac{1}{2}(Q(x, A) + Q(-x, A)) \geqslant \frac{1}{2}Q_1(-x, A)$. Thus, for $n \geqslant 1$,

$$\begin{aligned}
\tilde{Q}^n(x, A) &\geqslant \int_{y\in\mathbb{R}_+} \tilde{Q}(x, \mathrm{d}y)\tilde{Q}^{n-1}(y, A) \\
&\geqslant \frac{1}{2} \int_{y\in\mathbb{R}_+} Q_1(-x, \mathrm{d}y)\tilde{Q}^{n-1}(y, A) \\
&\geqslant \frac{1}{2^n} \int_{y\in\mathbb{R}_+} Q_1(-x, \mathrm{d}y)Q_1^{n-1}(y, A) = \frac{1}{2^n}Q_1(-x, A)\,,
\end{aligned}$$

where in the last equality we have used the result proven in the first part of this question.

Question 1.8. On what condition on Q_1 the measure π_3 will be the unique invariant measure for $\tilde{Q}$?

If Q_1 is irreducible (there is ν a measure on $\mathcal{B}(\mathbb{R}_+)$ such that for all $A \in \mathcal{B}(\mathbb{R}_+)$ such that $\nu(A) > 0$ and for all $x \in \mathbb{R}_+$, there is $n \in \mathbb{N}$ such that $Q_1^n(x, A) > 0$), then from the inequalities shown in the previous question we see that $\tilde{Q}$ is irreducible (relatively to the measure $\tilde{\nu}$ which is the extension of ν on the whole $\mathbb{R}$. In that case, we know that $\tilde{Q}$ admits a unique invariant measure and we have already verified that it is π_3.

Question 1.9. Propose a modification of the algorithm to sample from $\frac{1}{3}\tilde{\pi}_1 + \frac{2}{3}\tilde{\pi}_2$.

In the if statement sample from $Q(|X_k|, \cdot)$ is $U_k \leqslant 1/3$, otherwise sample from $Q(-|X_k|, \cdot)$

Algorithm 1 Input: $x_0 \in \mathbb{R}$

$X_0 = x_0$.
for $k \geqslant 0$ **do**
    Sample U_k, in an independent manner, with a uniform distribution on $[0, 1]$.
    if $U_k \leqslant 1/2$ **then**
        Sample X_{k+1} from $Q(|X_k|, \cdot)$
    else
        Sample X_{k+1} from $Q(-|X_k|, \cdot)$
    end if
end for